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MORRIS KLINE

Prediction of success in elementary algebra
GERALDINE SAX SHAW

An algebra program for the bright ninth grader
JOSEPH J. LATINO

THE NATIONAL COUNCIL OF TEACHERS

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Mathematics texts and teachers: a tirade

MORRIS KLINE, *New York University, New York.*

A sharp look at ourselves as teachers and the tools we use—textbooks—will prove beneficial. Do our textbooks follow patterns? What, if anything, does it signify?

TOO MANY TEACHERS, smugly convinced that they are offering fine presentations in classes and in texts, put the blame for our lack of success in the teaching of mathematics on the poor quality of the students. There is a student problem, but there are also three other factors which are responsible for the present state of mathematical learning, namely, the curricula, the texts, and the teachers. It seems to me that these factors have thus far evaded some well-warranted censure. I should like therefore to criticize the three factors. In turn, I suppose I can be criticized for just finding faults. However, not only are positive recommendations implicit in my criticisms, but a few explicit suggestions will be made at the end of the paper.

I would like to qualify the following remarks by pointing out that my experience has been in the field of college teaching. This experience does not warrant my setting myself up as an expert in this field, and it is all the less justification for writing with any authority about the high-school field. Moreover, it is my impression that the high-school teachers have done far more than the college teachers with experimentation and the evaluation of results. Nevertheless, I believe that most of the comments to be made, whatever their value, are applicable to both fields.

THE CURRICULA

It seems to me that at present too much time is being spent on arguing about details of specific curricula, while some broad principles are either being ignored or not sufficiently debated. Hence, I shall concentrate on the latter.

First of all, I wish to advocate the segregation of students in accordance with their future interests. I believe that such segregation is desirable at the high-school level and certainly necessary at the college level. The different capabilities and interests of the students are sufficiently marked at both stages to warrant the segregation appropriate to each. The high schools have already adopted this practice. Most colleges, however, are still bound to the practice of teaching the deadly college algebra and trigonometry to all students.

College curricula particularly are still in the medieval stage of development. The doctors of the Middle Ages treated every illness by bleeding the patient. We college teachers, analogously, give every student algebra and trigonometry and, unlike the doctors of old, definitely bleed the students to death. Even the students who come to us with a healthy interest in mathematics are given the same treatment, and we manage to kill them off too

Although these patients die one by one, we insist that we are improving their mental health.

It is rather interesting that the same teachers who favor algebra and trigonometry for all college freshmen behave somewhat differently when preparing to teach the calculus. In this case they generally do consider whether the students are going to be pure mathematicians, scientists, or engineers, and they prepare material accordingly. But they ask freshmen, regardless of whether they intend to be pure mathematicians or poets, to take the same course. Perhaps I should be fair and state that some colleges offer an alternative, a hodgepodge called general mathematics, which sins as badly as algebra and trigonometry, but commits the sins in more branches of mathematics.[1]

My next general observation on the curriculum is that in all courses we are still emphasizing meaningless technique, meaningless for all students. As prime examples in the college field I might mention Horner's method, the solution of oblique triangles by the laws of tangents and half-angles, and the solution of triangles, particularly the oblique triangle, by logarithms. These topics are dry, boring, and useless, and no arguments, not even "mental discipline," will ever make such material palatable or pregnant with significance. This "muscular" mathematics develops nothing. These techniques are not only dead but embalmed mathematics; they contain not a glimmer of life.

To me it seems obvious that techniques are the most boring and discouraging part of mathematics. But I have found a curious inability on the part of teachers to appreciate the barrenness of such material. I recently discussed this matter of techniques with a competent mathematician whose task was to take charge of the freshman program in one of our large universities. When I asked him whether freshmen courses in his university emphasize techniques, he said, "Oh, no, we teach the calculus." But when I rejoined that elementary calculus also contains a mass of techniques, he replied, "Well, we give definitions and theorems too." Such answers show how far teachers have gotten from understanding pedagogical problems. I suppose it would have seemed ridiculous to this professor if I had asked what the calculus had to do with the birth, loves, and death of man.

It must be admitted that some techniques are necessary. Mathematics does consist in part of processes which lead to new and useful knowledge. What shall we do about these necessary techniques? The answer is that they must either be applied as they are taught or taught where they are used in mathematics. From algebra, geometry, and trigonometry as such, nothing follows for the students. My thought here was best expressed by Alfred North Whitehead in an address made many years ago to a group of British teachers of mathematics:

> . . . elementary mathematics . . . must be purged of every element which can only be justified by reference to a more prolonged course of study. There can be nothing more destructive of true education than to spend long hours in the acquirement of ideas and methods which lead nowhere. . . . there is a widely-spread sense of boredom with the very idea of learning. I attribute this to the fact that they (the students) have been taught too many things merely in the air, things which have no coherence with any train of thought such as would naturally occur to anyone, however intellectual, who has his being in this modern world . . . the elements of mathematics should be treated as a set of fundamental ideas, the importance of which the student can immediately appreciate: that every proposition and method which cannot pass this test, however important for a more advanced study, should be ruthlessly cut out. . . .[2]

Of course mathematicians will say that this point of view excludes mathematics for mathematics sake. What, they may

[1] I would like to call attention to a proposed course for liberal arts freshmen, that is, students who do not intend to use mathematics in later life. See M. Kline, "Freshman Mathematics as an Integral Part of Western Culture," *Amer. Math. Monthly*, LXI (May 1954), 295–306.

[2] A. N. Whitehead, *Essays in Science and Philosophy* (New York: Philosophical Library, 1947), pp. 133, 139.

counter, about the beauty of mathematics? I believe that very few young people can appreciate the beauty of mathematics, and that we dare not rely upon this appeal to arouse and sustain interest in mathematics.

We must, then, apply the techniques we teach, as we teach them. Of course, many teachers contend that they make applications. But what applications? In algebra we teach students to mix brands of coffee so as to arrive at an undrinkable brew, and to mix cream and milk to the detriment of both. We ask students who cannot afford a swimming pool in their back yards to fill tanks. We use work problems to create pointless work. And we teach time, rate, and distance problems to students whose desire to go anywhere is not yet aroused. It is rather preposterous that the very teachers who teach the quadratic formula so fiercely and earnestly, expecting the students to be interested, could not themselves state one real use of it. The applications must be genuine and significant to the student. It is true that some significant applications can be found in mathematics books, but they are presented so briefly, with no preliminary explanation of the physical background, that they mean nothing to the student. For example, a formula for kinetic energy is given, and students are asked to calculate values of the formula. As far as the student is concerned, the formula might as well give the number of paved miles in Java. The applications I recommend do exist, but must be culled from various sources, and thus far we have not done so. Moreover, they must be presented with sufficient background material to make them meaningful to the students.

My next point on the matter of curriculum is that I object to the goal of functional mathematics on the college level. It is probably true that in the elementary and high-school curricula we must include such common practical tools as percentage, interest, area formulas, and the like. But beyond this there is little functional mathematics for the general student. What layman will ever use trigonometry in later life? What layman could possibly remember enough of his mathematics to compute an annuity premium? Almost nothing of what is properly college-level mathematics is functional. If this is a goal, then we had better teach electric wiring and house painting. That is functional for adult Americans.

A final point on the matter of curriculum. There is a strong movement afoot on both the high-school and college levels to teach modern mathematics. Apparently we must be up-to-date. And so we find current freshman texts presenting such topics as symbolic logic, postulational systems, Boolean algebra, set theory, rigorous derivations of the real number system, abstract algebraic theory such as groups and fields, and the like. This modern mathematics is much too difficult and, more important, thoroughly meaningless to the student. It is empty for them and indeed must be so. Only one who has thoroughly understood several concrete examples of a group can see the point in generalizing to abstract groups. How ridiculous to teach symbolic logic to students who confuse all A is B with all B is A! What can a postulational system mean to a student who does not appreciate the logical structure of Euclidean geometry?

I get the impression from reading many textbooks containing such modern topics that the authors have just learned this material themselves and are so pleased with their own accomplishments that they are impelled to display their knowledge. This remark may be unfair, but I cannot otherwise explain how they expect young people to see the significance of such abstract concepts.

It is, of course, proper to be modern. The subject of statistics looms large in today's world. And there is some elementary mathematics in this subject that can be absorbed by young people and be applied to genuine problems.

The above remarks on curriculum are, I know, rather disconnected. But all I intended to do was criticize some general practices and trends.

The textbooks

The most disturbing fact about mathematics texts is that they lack originality and repeat each other endlessly. A few thousand college algebra and trigonometry texts have been published since 1900. Practically all of these texts contain the same material; only the order of the topics is different. But there is hope for "progress" because each contains about ten topics, and the number of permutations of ten objects ten at a time is 3,628,800. It would be difficult to estimate how many trigonometry texts have been written that justify themselves because they treat the general angle before the acute angle. One can be sure, however, that just as many boast of treating the acute angle before the general angle. The only thing that is acute about these books is the pain they give the reader. Likewise, calculus books are now vying with each other on teaching integration before differentiation, after differentiation, or simultaneously.

Are there no variations among these books? There are variations such as the elementary college algebra and the advanced college algebra, the elementary advanced college algebra and the advanced elementary college algebra, the half-course and the full course, the 7/8 course, etc. Here, too, there is hope for "progress" because there are irrational numbers, and, hence, we can look forward to irrational algebra courses. We can expect the variations to run the gamut of the continuum.

What is especially disturbing about these books is that the authors are consciously dishonest to their profession. I spoke to one professor recently who had written "umpteen" trigonometries of the full and partially full type. When I asked him why he included such useless topics as the solution of oblique triangles by the law of tangents and the law of half-angles, he admitted that these topics were worthless, but said he included them because the books would sell better. Apparently, no matter how many trigonometries a man may write, not even one can reflect his honest judgment.

I asked another professor who published a stereotyped college algebra why he bothered to write such repetitious nonsense. "Oh," he said, "I can write the stuff between classes without having to think about it. Why shouldn't I do it?" Needless to say, no thinking was evident in the presentation of the material.

Another professor recently published a book and included some material that he believed to be undesirable. He admits this in his preface and then says quite candidly that he included this material with an eye to the market. Such honest dishonesty!

Those authors who repeat each other's material are certainly plagiarizing. But the plagiarism extends beyond that. One finds so many sentences, paragraphs, examples, and, of course, problems repeated from book to book. Paraphrases of whole sections of material covering many paragraphs are readily found. One author took over whole chapters from another book with only minor changes, acknowledging, of course, the inspiration of God, Euclid, Newton, and Einstein.

Quite clearly most textbooks are commercial jobs which make a contribution only to the authors' pocketbooks. The ethics of some college teachers, to say nothing of their mentalities, is evidently in a sad state. The only persons who can claim any credit for original work in connection with these books are the publishers' publicity men, who must think up good blurbs for the advertisements.

Let us, however, accept the fact that 99 per cent of the books are copies of each other and let us see what kind of job they do. After all, if the prototypes of these books were very good there would be some excuse for all of this blatant plagiarism. I

propose, therefore, to examine the quality of their exposition.

The first thing that strikes me about these books is the absence of any motivation. I should like to present a selection from a recent college algebra, which according to the publishers has sold well, which according to the advertising emphasizes ideas and principles, and which according to the preface presents reasonable motivation at the student's level. Of course I will not tell you the name of the author. Nor have I any fear that you will identify if from the selection, because the same selection could have been taken from any one of a hundred college algebras. Even if I gave you the page number, you could not identify it for the same reason. The selection is as follows:

Chapter XIII

Theory of Equations

96. The fundamental theorem of algebra. A polynomial equation in x or a rational integral equation in x of the nth degree is one that can be written in the form

$$c) \quad C_0x^n+C_1x^{n-1}+C_2x^{n-2}+\cdots+C_{n-1}x+C_n=0 \quad (C_0\neq 0)$$

where the coefficients $C_0, C_1, \cdots, C_n$ are constants and n is a positive integer. A root or solution of such an equation is a value or x which satisfies the equation; that is, a value such that the equation becomes a true statement when this value is substituted for x. To solve an equation means to find all its roots. Equations of degree 1, 2, 3, 4 are called linear, quadratic, cubic, and quartic, respectively. Thus far we have studied methods of solving equations of degree 1 and 2 only. In this chapter we shall consider the problem of solving equations of degree higher than 2. In the above equation, the coefficient of the highest power, C_0, is called the leading coefficient, while C_n is called the constant term.

The plunge which the above selection makes into the subject of the chapter has about the same effect on the reader as plunging him into a cold bath. It makes me shudder. I can hear the students saying, "We just managed to learn how to solve second degree equations without being told why we should bother with them, and now this author is talking nth degree equations. Even the police department stops with the third degree." Note that the one sentence that gives any clue at all to the object of the chapter is stuck in parenthetically towards the end of the paragraph. Why anyone should study nth degree equations is never mentioned. A mass of definitions is hurled at the student. No numerical example eases the learning of the notation, which baffles almost all the students.

It might be thought that the above example is extreme, but in fact it is entirely representative of practically all the writing in undergraduate mathematics textbooks. One would have to hunt extremely hard among these books to find a better presentation of subject matter. Why can't we motivate the topics so as to arouse some interest among the students? Even the prettiest woman seeks to enhance her appearance by dress and cosmetics.

Another criticism I must level at mathematical textbook writers is their inordinate pride in the brevity of their presentations, which I translate as incomprehensibility. Reasons for steps are often not given and where given are so enigmatic as to be almost useless. The authors of these books seem to be saying to the readers: "I have learned this material; now I defy you to learn it."

In their attempts to be brief and "smart" many writers use symbolism to an unnecessary and even repulsive extent. Especially in modern books one finds the symbol ϵ for "belongs to," $\leftrightarrow$ for an implication which goes both ways, $\exists$ for "there exists," and, of course, $\therefore$ for "therefore." Such symbols serve almost no purpose; the English language is better. The slight saving in space is more than offset by the psychological handicap that symbolism imposes on the students. They are scared of symbolism anyway and in addition are obliged to remember the meanings of these extra symbols. One can understand why such symbols are used in Russel and Whitehead's *Principia Mathe-*

matica, where the ambiguities of ordinary English must be avoided, but this value of the symbols is not utilized in undergraduate texts. As a just punishment the authors of these textbooks should be required to read the three volumes of the *Principia*. Perhaps it is sufficient to close this discussion of brevity with the remark that in mathematical writing brevity is the soul of witlessness.

Another fault in texts that is closely connected with the unnecessary profusion of symbols, is the sloppy handling of symbolism. Mathematicians pride themselves on their accuracy, their precision, and their clarity, and they profess to teach these virtues through mathematics. Indeed they defend the symbolism, which repels the student, as necessary to precision. But let us see how, in a few examples, textbooks use symbolism and notation. Let us compare the use of x in $3x=6$ and in $y=3x-6$. The two x's have almost nothing in common. In the first case x is some definite unknown the value of which we are seeking. In the second, x is a variable; it is not only not unknown, but can take on an infinity of values. Within some range we can assign values to it at pleasure. Certainly one can establish a connection between the two x's. One can and indeed does point out that the value of x in $y=3x-6$ which makes the left side 0 is the solution of $3x=6$. This thought is the link between equations and functions. But, conceptually, the two x's are quite distinct, and we want the students to keep the two situations distinct despite the fact that we may want to show their connection. It is true that usage has sanctioned the same symbol, x, in both situations, but should not textbooks warn students about this, especially in view of their boast about the unambiguity of mathematical symbolism? Let us note too that the equals sign has a different meaning in the two cases. In the first case it means numerical equality; in the second, y equals $3x-6$ by definition. Here, too, some remark about the different meanings of the equals sign should be made for the benefit of the students.

As another example, when dealing with $y=x^2$, say, most books will substitute 2 for x and then state $y=4$. If they were careful to couple the phrase $x=2$ with $y=4$ there might be little trouble, but when $y=4$ stands alone, as it often does, it confuses the student; 4 is just one value of y and not y itself. Quite often one sees a mess of y-values in the same context with no qualifications. Of course the student who is still learning about variables is confused.

A related ambiguity is the use of y to stand for the functional relationship and also for the dependent variable. A few books now use f for the functional relationship and y or $f(x)$ for the dependent variable.

In trigonometry we distinguish between conditional equations and identities, using $=$ and $\equiv$ for the respective cases. Why don't we use the identity symbol in $a^2-b^2=(a-b)(a+b)$?

Most calculus books still use dy/dx for the derivative. The difficulties which this notation creates are well known, but they apparently do not deter brave authors.

A very common error in trigonometry books is the statement $1^\circ=\pi/180$ radians. Since the equals sign means numerical equality and since equations relate numbers only without regard to what they stand for, this statement is nonsense. The correct statement is that the number of degrees equals $180/\pi$ times the number of radians. Likewise, such statements as 1 yard = 3 feet are incorrect.

Another point about the quality of the exposition is the writing itself. It seems clear that many authors profited as little from their study of English as their students profit from the study of mathematics. Dangling participles are a dime a dozen. Long, awkward, graceless sentences are the rule. In manuscripts I have even seen common spelling errors, but apparently the editors catch these. Clichés are common. For example, "It might be well at this point to discuss. . . . " Why? The

phrase avoids thinking through a valid reason for introducing the new idea. "This subject is beyond the scope of the text," is a very uninformative, discouraging, and impertinent way of terminating a subject. The piling up of such clichés has been labelled the bricklayer approach to writing.

The indifference to good writing is exemplified by the attitude of one professor I have met. This professor, author of several textbooks, is aware that sentences are constructed in accordance with rules of grammar, but, like elementary school pupils, he is uncertain of these rules. In speaking he would say, for example, "That's what Jones told him and," and, after a long pause, he would conclude, "I." When chided for such mistakes, this professor would reply quite seriously, "Well, my wife knows the grammar"!

My final point as to style is that textbook writers seem to share the belief that scientific writing must be objective and impersonal. As a consequence, the writing is cold, spiritless, mechanical, and dry. The books have no authors. They are not only printed by machines, but written by machines.

We must not forget to pay our respects to the publishers. There is no doubt that they have contributed to the sad state of textbooks. Because more money can be made and less risk taken in putting out a standard text they seek these out. Also, because they are interested in profit they will take a poor book if they know that it will be used by a large university. They are also guilty of poor judgment. They ask readers to advise them on manuscripts and pay so little for the advice that the readers cannot afford to do a good job. They also are guilty of misrepresentation. In a recent rather expensive advertisement of a college algebra book the publishers prepared a booklet and reproduced pages of the text. The text proper was the same dull mechanical material that can be found in countless other college algebras. But 3 of the 400 and some pages in the book did discuss the values of algebra. These were the pages reproduced.

The teacher

It is painful but necessary to become somewhat personal and consider the teachers themselves. If it is still possible for a teacher to express an opinion, I should like to affirm that the most boring group of people in the current era are the college mathematics teachers as teachers. The quality of mathematical teaching is inversely proportional to the significance of the subject, and the proportionality constant is near zero, if not negative. Insofar as teachers reveal themselves by what they put into their textbooks as worthwhile subject matter, I believe the above condemnation has been justified. But there is additional support for this opinion which comes from contact with the teachers themselves.

I believe that the bulk of the college mathematics teachers can be classified into three groups, which I should like to describe. The first group consists of the research men who teach undergraduates. These men look down upon teaching as an art, sneer at the mere teacher, act condescendingly toward the freshmen whom they regard as dull and worthless, and complain of the chore of teaching undergraduates. Such research men may know the mathematical theory of ideals, but they certainly are not familiar with the ideals of teaching. Why do intelligent men behave this way? To some extent they are forced into this position. The universities pride themselves on the research done under their sponsorship and hence influence research men to emphasize this activity. Indeed, many universities are hothouses of research maintained at fever temperature. Moreover, promotions and increases in salary are most often based on research. Good teaching, on the other hand, is hard to judge and rarely recognized. It is no wonder, therefore, that men who are capable of doing research concentrate on this activity.

Why then, do research men teach undergraduates? It is because they cannot fill the required number of hours of teaching with graduate work only. There are not enough graduate students, and even if there were it would be difficult to teach three or four graduate courses.

The second type of college teacher of mathematics is best described as the stagnant undergraduate teacher, the one who knew little when he began to teach and forgets more and more as the years go by. He teaches only algebra, trigonometry, analytics, and calculus. These people went to college once upon a time and learned how to perform certain mathematical techniques, just as a carpenter or machinist learns his trade. They repeat the processes like marionettes. The typical undergraduate teacher has no more to recommend him than his ability to remember a set of procedures, which he then teaches others to perform. Steeped only in mechanical techniques and unpossessed of insight, these dull teachers of mathematics ply their trade with proficient ignorance.

Such teachers naturally resist any changes in curricula because new curricula generally involve learning new material. They therefore argue that what was good enough for great grandfather is good enough for their students. Do these teachers even consider varying their teaching? Oh, yes! They debate hotly whether to put partial fractions before the binomial theorem or after.

These undergraduate teachers know nothing about history, science, philosophy, or art even insofar as these subjects relate intimately to their daily teaching. They know still less about the broad ideas which underlie the mechanical procedures or the ultimate objectives at which these procedures aim. When told that they should discuss these ideas and objectives and occasionally at least relate mathematics to other fields of knowledge, they are shocked. They have heard that Plato advocated the teaching of mathematics because it trains the mind, and that is argument enough for teaching the usual material. They therefore hide behind the facade of symbolism and techniques that, because they are new to the student, bewilder him. The student is awed by this display of seeming profundity and credits the teacher with a knowledge and intelligence he does not possess.

Because their "knowledge" is limited to technical processes, these teachers almost never discuss any ideas in class and in fact are totally deficient as public speakers and in their command of language. When pulled away from a presentation of purely formal techniques, they become frightened and rush back into the familiar routines.

There is, of course, a more than intimate connection between the quality of these teachers and the predominance of the mechanical textbooks discussed above. Occasionally a text appears that discusses to some degree the ideas underlying the manipulative mathematics. But the standpatters will have nothing to do with such books. They call this type of book verbose. But the real reason for their objection emerges as one talks to them. Sooner or later they say, "Well, what's the teacher going to do if the book tells the student everything?" One sees from this remark what these teachers mean by teaching. They prefer to use an inadequate and even incomprehensible text, so that they can fill in the steps in class and save the day for the student. The sterility of mathematical teaching may definitely be charged to the ignorance and narrowness of these stagnant undergraduate teachers.

When these men and women are faced with the inevitable consequence—that the students are bored and baffled by meaningless techniques—they criticize the students for lack of interest. Actually it is such teachers who are not interested. By continuing to teach, however efficiently, the same old techniques, which they have at their fingertips by constant repetition, they are just taking the easiest course and are avoiding the thinking that they profess

to be inculcating in their students through the study of mathematics. They are the practical men who practice the errors of their forefathers.

I should like to relate a couple of remarks by teachers which may sound like exaggerations but which, I believe, reflect their true attitudes. I knew a teacher who taught the standard undergraduate material for years. Once in a while, the department would consider a new text for a course. Then this teacher would be sure to object with the remark, "But this means doing new problems." Another teacher used to argue against any changes in curriculum by pointing to the fact that he had taught the same material for 25 years and therefore was best qualified to judge the worth of a topic and how it should be taught.

The caliber of these undergraduate teachers has been well described by Scott Buchanan in his *Poetry and Mathematics:*

> Mathematics suffers much, but most of all from its teachers. As a result of bad pedagogy—and I mean the kind often judged best by administrative pedagogues—the appearance of an algebraic formula, a geometrical figure, or an innocent set of symbols reduces the reader to an unbecoming attitude of hypocritical humility. A great many sometime students of mathematics try to persuade themselves that they haven't mathematical minds, when as a matter of fact they have had only non-mathematical teachers. Mathematics is not what most teachers of mathematics teach. They, with the good intentions of conveying what they themselves have only as a skill of manipulation, have unconsciously worked hocus-pocus on their pupils. They have repeated and illustrated opaque formulae, sometimes to the admiration, but almost always to the bewilderment, of their students.

One might think that such teachers with limited backgrounds and no incentive to advance their own knowledge of mathematics would ultimately be called to account by their chairmen. But I have heard many chairmen argue that a knowledge of undergraduate mathematics is just right for the teachers. In fact "anyone who knows calculus can teach freshmen." I have read that high-school chairmen are often no better. Ad after ad appears for teachers of mathematics and physics which read: "Wanted, man for football (or basketball) coach. Must teach mathematics."[3]

I shall conclude this characterization of the stagnant undergraduate teacher with the observation that one cannot produce Rembrandts by teaching house-painting, particularly, if one is a house-painter. Socrates was condemned to death for supposedly degrading the morals of the youth of Athens. What punishment should be meted out to teachers who degrade mathematics, cause the students to hate the subject, and in many cases poison the student's mind against all learning?

The third type of undergraduate teacher is what one might call the "cheap help" type. Many so-called universities turn the bulk of their undergraduate teaching over to graduate students. The real reason for this practice is to save money, but the reason offered publicly is that the graduate student receives support. This practice is criminal. The graduate student may know the processes he is asked to teach, but he cannot be too well educated at that age; he cannot be mature in his handling of student problems; and he cannot be interested in teaching at a time of his life when he is concerned with getting an advanced degree. Of course, the undergraduate students are cheated. I have heard many chairmen argue that these young people are more enthusiastic than the older teachers. This is probably true, but the argument serves only to condemn the older teachers.

Are there no capable, sincere, hard-working college teachers? Yes, of course, but these are few in number, and they are not only not encouraged, but discouraged. Deans and presidents pay lip service to good teaching, but do not recognize it. Under the usual pressures of administration, they favor research people, who attract more attention to the institutions

[3] See *Mathematics Magazine*, 28 (Jan.-Feb. 1955), 183.

and who, because their work is published, get offers and are in a position to bargain.

Some positive suggestions

My remarks thus far have been negative and hence subject to the criticism that any fool can say what is wrong; the real need is to offer positive suggestions. I do not agree with this criticism. We have not been frank enough in stating the deficiencies in our curricula, our texts, and our teachers. We have recognized that mathematics education is not reaching the students and is losing its traditional place in the high schools and colleges. But, perhaps out of courtesy and politeness, we have hesitated to attack the real weaknesses. Hence, I believe that an honest discussion of our faults is in order here and elsewhere. For example, if those of us who review texts in The Mathematics Teacher, *School Science and Mathematics*, *The American Mathematical Monthly*, and other journals would soundly condemn the 2,000th repetition of a college algebra book instead of saying that it meets the usual standards, we might discourage such writing.[4] However, I would like to advance some positive suggestions. Let me reverse the order of the major topics I have considered.

First, as to the teacher, we want more recognition from administrators for the good teacher. This step is no doubt contingent upon the schools getting more money to pay teachers, but let us at least clamor for rewards for the good teachers.

I would also urge every teacher to become an actor. His classroom technique must be enlivened by every device used in theater. He can be and should be dramatic where appropriate. He must not only have facts, but fire. He can utilize even eccentricities of behavior to stir up human interest. He should not be afraid of humor and should use it freely. Even an irrelevant joke or story perks up a class enormously.

The teacher should solicit and answer all kinds of questions, even irrelevant ones, to encourage active participation in the work. Many teachers are worried that too many questions take up time, and keep them from covering the required topics. But what is the use of covering ground if one is getting nowhere?

With respect to textbook writing it should be unnecessary to urge prospective authors to learn something about writing, but I have met many teachers who jumped directly into the writing of texts without any previous writing experience. In the writing of mathematics proper the author should remember to motivate each new process or theorem; he should discuss the rough idea first by means of intuitive, concrete examples; he should give the historical setting if it is enlightening or even interesting; he should give examples of where the theorem or process does not work; he might suggest the wrong theorem and then modify it. In brief he should "create" the mathematics as he develops it. Of course, in presenting the final version he should explain all steps carefully and discuss the significance of the result.

I should like to urge too that mathematical writing be more human. Prospective textbook writers should read some of the works of the great scientists, for example, Pascal, Kepler, Galileo, and Newton. Let me take a moment for what may be an extreme example. In 1606 Kepler wrote *De Stella nova in pede Serpentarii* in which he undertook to explain and discuss the appearance and disappearance of a new star. In one passage he writes, "If I should be asked what will come to pass; what is it that this apparition forebodes? I shall answer without hesitation: First, a flood of writings published by numerous authors, and much labor for the printers." Later he remarks that the blind forces of the atom had nothing to do with the appearance and disappearance of the star and says that his wife, Barbara Müller

[4] The March 1955 issue of *The Amer. Math. Monthly* contains a review of four "standard" college algebra books published in 1954. The reviewer clearly recognizes that these books have little new to offer, but courteously confines his comments to the differences among these four books.

(who was not a scientist) had the same opinion. Then he tells us that one day he sat down to eat a salad prepared by his wife, and he asked her, "If tin plates, lettuce leaves, grains of salt, drops of oil and vinegar, and fragments of hard-boiled eggs had been floating in space ever since the creation, in every direction and without order, could chance have brought them together today to form a salad? Not so good a one, I am certain, replied my fair spouse, nor so well made as this one." There are many glorious literary passages in Kepler's numerous scientific writings, but I have chosen one which, seemingly trivial, illustrates the human spirit of his writing.

Finally, one major thought as to curriculum. As mathematicians, we would like to present the deductive, pure mathematics that we regard as the essence of the subject today. But this is not the material that will arouse the interest of the students. It is the goal, but not the approach. We are so familiar with the subject that we no longer appreciate the degree of its abstractness. Let us recall that of the hundreds of civilizations which have flourished, only one developed the quality and kind of mathematics that we respect today, namely, the Greeks. The point of the remark is that abstract, deductive mathematics is, if history is a guide, not natural. It is distant from the thoughts, interests, and activities of the normal human animal. It is a highly sophisticated, difficult, and even esoteric subject. Let us therefore not expect too much from the students.

How should we approach mathematics so as to arouse interest, keep the subject meaningful, and yet ultimately have the students appreciate the power and beauty of abstract deductive mathematics? On the basis of a study of the history of mathematics as a whole and of the approach made to mathematics by even the greatest minds, Archimedes, Newton, and Gauss, I believe that the answer is to tie mathematics to the study of the physical world. I do not mean that mathematics should be buried in some corner of a physical science course, but rather that we should motivate and interpret mathematics in terms of genuine physical problems. Mathematics derives from the study of nature and is valuable mainly because of what it returns to nature. I would like to use the words of Alexander Pope to express this philosophy:

> First follow nature and your judgment frame
> By her just standard, which is still the same.
> Unerring Nature, still divinely bright,
> One clear, unchanged and universal light,
> Life, force, and beauty must to all impart.
> At once the source, and end, and test of Art.
> . . . . . .
> Those rules of old discovered, not devised,
> Are nature still, but nature methodized;
> . . . . . .
> 'Tis Nature's voice, and Nature we obey.

Perhaps if we approach mathematics through the study of nature we shall nevertheless arrive at the abstract, deductive truths we wish to teach, and some will grasp the true value and nobility of our subject. Other students may see no more than the bright colors they see in great paintings, but at least the color of our subject will impress them and cause them to respect it.

Apologies

I should like to apologize if I have slandered and shocked people, though shocks have been known to restore sight to the blind and hearing to the deaf. I hope I have not been trite. Many discussions of teaching problems voice nice sentiments and state broad objectives, but do not get down to concrete difficulties.

Prediction of success in elementary algebra

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An account of the experiences that one school gained while trying to answer the ever-present question, "Who should take algebra?"

TEACHERS of high school mathematics have long been concerned with the high percentage of failures among students of elementary algebra. Although new methods and objectives may help to reduce the large mortality, the problem still persists. This study is concerned with the quest for a reliable method of forecasting the success of a student of elementary algebra. The importance of this search lies in its close association with other related problems. There is, for instance, the problem of frustration that comes as the result of failure. There is the problem of improper apportionment of homework time, with undue monopoly of time by a subject in which a student is due to fail anyway, and neglect of other important studies. There is also the problem of too early school-leaving due to disheartening school experiences.

It is not to be inferred that all these problems will be solved by the formulation of an accurate system of predicting a student's success. Nevertheless, a good method of forecasting a pupil's chances for passing or failing elementary algebra should do much to eliminate possible traumatic experiences for him.

The quest for a successful prediction device is a difficult one. There is no mathematical crystal ball with which success or failure can be predicted accurately. A number of approaches have been attempted. Many educators believe that success is not beyond anyone with a certain minimal intelligence. In line with this theory, Sister Mary Rosilda[1] studied the relation between IQ and scores on a first-year algebra test for 635 ninth-grade pupils, finding a correlation coefficient of only .42. Others feel that there is a special aptitude for algebra which the student either has or has not. Various algebra tests have accordingly been devised, such as the Iowa Algebra Aptitude Test.[2] Recognizing its limitations, Kraft reported a modification of this procedure[3] in which scores on this test were used to classify pupils into "surely," "maybe," and "no" groups; and evidence, such as previous success in school, was used to move students from the "maybe" into one of the other groups. Still another approach is through a study of reading ability. Since so much of elementary algebra is concerned with verbal problems requiring reading comprehension before algebraic principles can be applied correctly, the score on a reading test might be an aid in prediction procedure. Fay[4] determined abilities of sixth-grade pupils and found that those who were superior in reading

[1] Sister Mary Rosilda, "Is an IQ an Index to Algebra Ability?" *Journal of Educational Research*, Vol. 44 (January 1951), pp. 391–393.

[2] H. A. Greene and A. H. Piper, *Iowa Algebra Aptitude Test* (Iowa City, Iowa: 1942) Educational Research and Service, Extension Division, University of Iowa.

[3] Ona Kraft, "Methods Used in the Selection of Pupils for the Study of Algebra and Geometry in Cleveland," THE MATHEMATICS TEACHER, Vol. XLIX (May 1946), pp. 236–239.

[4] Leo C. Fay, "The Relationship between Specific Reading Skill and Selected Areas of Sixth Grade Achievement," *Journal of Educational Research*, Vol. 43 (March 1950), pp. 541–547.

skills had no higher achievement in arithmetic than those who were inferior. Previously Eagle[5] had conducted a study in which he found no evidence that improvement in general reading comprehension or general vocabulary would increase proficiency in eighth- and ninth-grade mathematics.

Each of the studies mentioned has considered only one factor in its relationship to success in algebra. In this study the factors of IQ, reading ability, and "algebra aptitude" were studied in various multiple combinations, in addition to being regarded as isolated factors, in an attempt to produce greater accuracy of prediction.

Although the avowed purpose of this study was to find a better method of forecasting by the use of the aforementioned tests it was important to realize in advance that this might be an impossible task, because so many unmeasurable factors affect the pupil's success. Every algebra teacher has known the student whose drive pushes him to a passing grade despite intellectual handicaps. On the other side of the ledger are the all too frequent cases of the defeatist attitude—"I'm just like my father. He never was any good at math either"—which leads the student to abandon all effort and resign himself to failure despite his possible ability to succeed. As Wesman[6] says, "The importance of interests and other personality traits for learning skills or acquiring knowledge needs no exposition. How well a person will acquire proficiency depends so much on his interest in his task, on his drives and goals, that the layman appreciates these conditioning factors as thoroughly as does the psychologist." However, in the large school selected for this study, the personality factors mentioned could not be weighed, due to the limitations of time and personnel. The study, therefore, had to proceed on the basis of objective tests.

The sample used was a group of 387 freshmen attending Sewanhaka High School in Nassau County, New York, which draws its students from several suburban communities with a wide socio-economic range. Under the guidance of Miss Ida Ostrander, head of the mathematics department, methods are being constantly revised and improved with the aim of making teaching more efficient. Both aptitude and achievement tests are administered to facilitate evaluation. All students, in line with the enlightened and progressive administration of the entire school, also undergo certain other standard tests given by the guidance department.

The group used in this study is composed of those algebra students to whom the following tests have been administered: Iowa Algebra Aptitude Test[7] at the end of the eighth year, the Otis QS Mental Ability Test (Gamma)[8] at the beginning of the ninth year, and the Iowa Silent Reading Test,[9] also at the beginning of the ninth year. Two criteria for success at the close of the year's course are a departmental final examination devised by the mathematics department, and the Lankton First Year Algebra Test.[10] Of the group, 64 pupils failed the first term's work. Although important data about them has been included in the study, these pupils cannot be included in the correlation calculations because the criteria for success could, of course, not be applied to them. Obviously, since they could not proceed to the second term of

[5] Edwin Eagle, "The Relationship of Certain Reading Abilities to Success in Mathematics," THE MATHEMATICS TEACHER, Vol. XLI (April 1948), pp. 175–179.

[6] Alexander G. Wesman, "What Is an Aptitude?" Test Service Bulletin, No. 36, The Psychological Corporation (August 1948), p. 2.

[7] *Iowa Algebra Aptitude Test, op. cit.*

[8] Arthur Otis, *Otis Quick-scoring Mental Ability Test* (Gamma test for high schools and colleges (Yonkers, New York: The World Book Company, 1939).

[9] H. A. Greene, A. N. Jorgensen, and V. H. Kelley, *Iowa Silent Reading Test* (Yonkers, New York: The World Book Company, 1943).

[10] Robert Lankton, *Lankton First-Year Algebra Test* (Yonkers, New York: The World Book Company, 1951).

algebra, they could not take the achievement test at the end of the year. Consequently, their success could not be correlated on a quantitative basis with the three tests used for prognosis.

The median IQ of the entire group was 115, in contrast to 100, the average for the standardizing population for the Otis IA QS. Median score in the Iowa Algebra Aptitude Test was 59, equivalent to the seventy-third percentile of the standardizing population. The superiority of the group in these two respects is probably due to the fact that the sample had already been partially selected. Those whose percentile ranks in the Iowa Algebra Aptitude Test were below 40 were advised not to take the course. There was no compulsion about this, and, as will be evident from the data to follow, many students disregarded this advice because of collegiate ambition or parental prodding. Nevertheless, it must be borne in mind that the results were influenced by the fact that only those who studied algebra were considered, and the important group who may have been dissuaded from doing so, on the basis of the aptitude test marks or previous difficulty in mathematics or lack of desire to prepare for college, must be omitted.

In this study it was assumed that the tests were administered under favorable conditions and scored accurately. It was not assumed that reading ability, intelligence, and algebra aptitude are mutually exclusive factors, or are the only factors, or are, in fact, factors at all in the successful completion of a course in elementary algebra. Nevertheless, an attempt was made to correlate the three predictive measures with degree of success, using multiple-screen and multiple-correlation methods. The coefficients of correlation were computed by the product-moment method from the correlation tables.

After a careful study of the data, it was decided to use the Lankton test as the criterion for success, rather than the departmental final examination. Because this test was completely objective insofar as scoring was concerned, was not specifically "crammed" for by the students, and covered the scope of the course, it was considered good for the purposes of this study.

Table 1, showing the high percentage of failures, serves to confirm the pressing need for studies of this sort. Almost three out of every ten students of elementary algebra failed the course, indicating the importance of finding these students *before* they have started the study of algebra.

Even the simple frequency distributions set up as preliminaries to further investigation indicated the difficulties to be encountered in finding individual patterns.

TABLE 1—PERCENTAGE OF FAILURES[11] IN 387 STUDENTS OF FIRST-YEAR ALGEBRA

	Number of pupils	Percentage of pupils
Failed first term	64	17%
Failed second term	48	12%
Total failures	112	29%
Passed	275	71%
Total	387	100%

From the consideration of the scores on the separate tests, Iowa Reading, Iowa Algebra, and Otis, it was determined that while group trends are clearly evident, many individuals do not conform to the pattern. Most of the students with high reading scores passed algebra; yet 5 out of 14 students who read on a fifth-year level passed the course, and 7 out of 31 students who read on an eleventh-year level failed the course. Similarly, although most students' success in the algebra aptitude test indicated a good prognosis, 8 students with scores over 70 (eighty-seventh percentile or higher) failed algebra, and several who scored low in the aptitude test passed algebra. There were failures even among those who ranked as high as 132 in

[11] Thirty-fifth percentile or below on Lankton test was considered to be failing for purposes of this study.

IQ; and several students with IQ's in the middle 80's were able to pass.

The implication drawn from the distributions of marks is that whatever low scores are set as entrance barriers for elementary algebra, using any one of these three tests, some students who might perhaps succeed would be eliminated. Statistically speaking, the proportion is small. Nevertheless, humanly speaking, if even one student is unjustly deprived of his chance, the system has failed.

The failure of accurate prediction on the basis of any single test leads inevitably to the question of whether greater success could be achieved on the basis of the scores of two or more of the tests, taken in various combinations. Many multiple screening devices were attempted. Two which came closest to an acceptable pattern are summarized in Table 2.

From the viewpoint of group predictions, these screening plans appear at first glance to be successful. However, prediction for any given individual is inadequate. Plan I, it is true, would have made it possible to eliminate 38 of the students who were doomed to fail algebra. Nevertheless, 24 students who did pass the course would have been unjustly eliminated along with the others on the basis of the scores selected, and 74 students who did fail would not have been picked up in the screening.

Similarly, Plan II, while it missed only 46 students who eventually failed, would have unjustly eliminated 56 students along with the 66 who did fail.

Other screening attempts ended in even greater failures insofar as individual prediction possibilities were concerned. The students who would have been eliminated by the use of either of the multiple screening methods mentioned above obtained scores ranging up into the 90's in the Lankton test. In other words, they were not just borderline cases. This probably could have been seen by an inspection of the individual records of the pupils. For every set of scores of a student who ranked low in the Lankton test, there are similar, if not identical, patterns in IQ, Algebra Aptitude, and Iowa Reading tests among those who did well in the Lankton examination.

Correlation computations ended in failures similar to screening devices, insofar as individual predictions are concerned, although positive correlations indicate some relationship of scores in the various tests administered. These correlations are summarized in Tables 3 and 4.

TABLE 2—Results of Two Plans for Multiple Screening

Cutoff scores	Eliminated		Total eliminated	Misses
	Passed	Failed		
Plan I				
IQ under 93	4	15	19	97
Iowa Algebra Aptitude, 30 or less	5	12	17	100
Iowa Reading, under 7.0	20	24	44	88
Composite total	24	38	62	74
Plan II				
IQ under 93	4	15	19	97
Iowa Algebra Aptitude, 30 or less	5	12	17	100
Iowa Reading, under 8.0	54	62	116	50
Composite total	56	66	122	46

TABLE 3—Correlation of Four Tests to Each Other
Based on Results for 323 Students Who Completed First-Year Algebra

	Lankton Algebra	Otis QS IQ (Gamma)	Iowa Algebra Aptitude	Iowa Reading
Lankton Algebra	—	.53	.54	.45
Otis QS IQ (Gamma)	.53	—	.43	.45
Iowa Algebra Aptitude	.54	.43	—	.38
Iowa Reading	.45	.45	.38	—
Departmental Final Exam	.60			

As measured by the tests in this study, intelligence is correlated with achievement in algebra .53; and reading ability is correlated with algebra achievement .45. The Iowa Algebra Aptitude test for this particular group correlated with achievement .54, negligibly better than the intelligence test. For this group then, the time and labor involved in administering the aptitude test would have produced only slightly better results for forecasting than the already available IQ scores. The aptitude test would be exceedingly useful, of course, in the analysis of areas of difficulty and in other ways not considered in this study. A marked relationship between the aptitude test, the reading test, and the intelligence test, considered as a triple factor, and the achievement test was indicated by .77 correlation, a relationship which could not be accounted for by chance.

The correlations indicated in Tables 3 and 4 are all high for group prognosis, especially in consideration of the high selectivity of the sample, previously referred to. Had it been possible to enroll the entire freshman class of the high school in elementary algebra, with the greater range of intelligence, reading ability, and aptitude for algebra, an even higher set of correlations would have been the result, according to the laws of probability. Thus, there is justification in making group predictions as to percentages of the groups which will probably pass or fail. There is also justification in concluding that students' marks in the Algebra Aptitude test, the Otis intelligence test, and the Iowa Reading Test are apt to be highly correlated to each other in various combinations, that a student's mark in the Algebra Aptitude test is apt to be a better indicator of his possible success in algebra than his mark in the reading test, and very slightly better than his Otis IQ. It might also be concluded that the three factors, reading ability, intelligence, and algebra aptitude, as measured by these tests, are very closely related to success in algebra, as measured by the Lankton test, when they are considered as a combined factor. However, all these facts and figures still do not provide the basis for making definite accurate prognoses for individual pupils. Weighting the scores on the three prognostic tests provides slightly better accuracy, but still is not sufficiently accurate to justify the involved computations.

Summarizing the conclusions of this study, it has been shown that, on the basis of the scores of 387 high school freshmen students of elementary algebra, the Iowa Algebra Aptitude test, the Otis IQ QS

TABLE 4—Multiple Correlations
Based on Results for 323 Students Who Completed First-Year Algebra

Lankton and Algebra Aptitude correlated with IQ	.56
Lankton and Iowa Reading correlated with IQ	.58
Reading and Algebra Aptitude correlated with IQ	.53
IQ, Iowa Algebra Aptitude, and Iowa Reading correlated with Lankton Algebra test	.77

test, and the Iowa Reading test are, in descending order, good indicators of group success in algebra. A greater degree of success of prediction is achieved by using the combined scores of the three tests. A high degree of correlation is indicated among these factors. However, none of these relationships, singly or in combination, whether by correlation or screening methods, is sufficiently high to warrant definite prognosis for individual students.

Two areas for further investigation are suggested by the above findings. The first is the use of cutoff scores and multiple screening, not to eliminate students, but to put the spotlight on those who are apt to have difficulty. Instead of being barred from algebra classes, these students could be given extra help to insure their understanding of the work, or could be placed in special classes to proceed at a slower rate. In this way both students and teachers would profit by early recognition of trouble spots. Moreover, those students whose drive would lead them to overcome their deficiencies would not be banished on the basis of predictive test results. However, even while work is being done to help the borderline pupil, ninth-grade algebra itself and its methods must be constantly re-evaluated and revised in the light of the changing world at large and the changing local community of any particular school. Algebra thus made more meaningful for the students should incidentally be easier for them to pass.

A second area for further research is indicated by every shortcoming of standardized predictive tests. Each pupil must be considered as an individual, as a complete person, rather than as a list of test scores. The correlations obtained in this study indicate that the tests used are indeed extremely useful and important. Nevertheless, unless they are supplemented by a knowledge of the interests, attitudes, and personal emotional adjustment of each student, all such tests must fall short of their goal of perfect prediction.

What's new?

BOOKS

College

Analytic Geometry, Fifth Edition, Clyde E. Love and Earl D. Rainville, New York, The Macmillan Company, 1955. Cloth, xiv+302 pp., $4.00.

Trigonometrical Series, Second Edition, Antoni Zygmund, New York, Dover Publications, 1955. Paper, 329 pp., $1.85.

Vector Analysis, Homer E. Newell, Jr., New York, McGraw-Hill Book Company, 1955. Cloth, xi+216 pp., $5.50.

Miscellaneous

Free and Inexpensive Learning Materials, Nashville, Tennessee, George Peabody College for Teachers, 1956. Paper, viii+244 pp., $1.00.

Matter and Light, the New Physics, Second Edition, Louis DeBroglie, New York, Dover Publications, 1955. Paper, 300 pp., $1.60.

Piloting and Navigating in Coastal and Inland Waters, E. M. Savage, San Jose, California, Marine Publications, 1955. Cloth, 198 pp., $5.00.

Reflections of a Physicist, Second Edition, P. W. Bridgman, New York, Philosophical Library, 1955. Cloth, xiv+576 pp., $6.00.

The Common Sense of the Exact Sciences, Second Edition, William Kingdon Clifford, New York, Dover Publications, 1955. Paper, xiii+249 pp., $1.60.

BOOKLET

The Superior Pupil in Junior High School Mathematics (Bulletin 1955, No. 4 of U. S. Office of Education), Superintendent of Documents, U. S. Government Printing Office, Washington 25, D. C. Booklet written by Earl M. McWilliams and Kenneth E. Brown, 57 pp., 25¢.

An algebra program for the bright ninth grader

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What to do about the bright student in the mathematics classes is always a perplexing question. Here are some suggestions for your consideration.

AT TEACHERS' CONFERENCES these days perhaps no other query arises more often than the one which asks, "When are we going to do something for the bright pupil?" It is not a new question. In 1940 the National Council in its *Fifteenth Yearbook* discussed the "Problems of Retardation and Acceleration."[1] It was noted that Terman had found 20% of the school population in the superior group with IQ above 110. Terman's figures are substantiated by a study of the intelligence distribution of 2,241 children who entered the eighth grade in Rochester, New York, September, 1938.[2] The yearbook noted that "these early findings have been confirmed by many later studies, including others by Terman."[3]

William D. Reeve in the January, 1955, issue of THE MATHEMATICS TEACHER attempted to present the need for a new national policy and program in secondary mathematics. He noted, "In attempting to sugar-coat the course in mathematics for the slow-learning students, we are at the same time lowering the standards of accomplishment for the more gifted, and as a result the bright student becomes the most retarded of all."[4] These two citations will serve to remind us of the omnipresence of the problem of the bright child.

What is holding us back? The question contains a false premise. There are some strides being made to meet the needs of the bright student. Some of these measures include differentiated assignments, mathematics clubs, and projects involving the construction of things that require mathematics or that exhibit principles of mathematics. The *Fifteenth Yearbook* made excellent recommendations for the accelerated pupil. But as Reeve noted, "It would be interesting to know how many teachers in this country have never heard about, much less read, the two or three outstanding national reports in recent years."[5] There is a strong suspicion that not many of us have bothered to investigate the literature available on the subject.

What is the reason for this? It is difficult to say. But some factors are discernible. In a survey made by the writer the following conclusions seemed to be true:

1. Out of thirty-two ninth-grade mathematics teachers in thirty-two different

[1] "The Place of Mathematics in Secondary Education," The *Fifteenth Yearbook* (The National Council of Teachers of Mathematics, Washington, D. C., 1940), Chapter VII, pp. 120–147.

[2] *Ibid.*, pp. 122–123.

[3] *Ibid.*, p. 123.

[4] William David Reeve, "The Need for a New National Policy and Program in Secondary Mathematics," THE MATHEMATICS TEACHER, XLVIII (January 1955) p. 6.

[5] *Op. cit.*, p. 3.

school systems, none indicated that they had an entirely different program for the bright pupil.

2. Only one of the teachers seemed unconcerned about the problem.

3. Seventeen of the thirty-two indicated that at various times they give the bright student an occasional problem or two in lieu of or in addition to the regular assignment.

4. Thirty-one of the teachers thought that the modern ninth-year algebra text made adequate provisions for the bright student and that the mathematics contained therein was as much as the bright ninth grader could absorb.

5. All teachers seemed agreed on one thing. With today's pupil loads and other clerical duties, provision for the bright student is often impossible.

While there may be contradictory elements in these answers, the general observation might be made that this group of teachers feels that they should be doing something for the bright pupil, that the textbook has the available material for the bright pupil, but that time and load prevent them from doing this aspect of their work well. Now, although this is not an authoritative study, it is a straw in the wind.

A ninth-grade mathematics teacher is limited by the time available for the bright student. There is the eternal job of getting as many through as possible. Also, a ninth-grade mathematics teacher is limited by lack of adequate practical techniques for meeting the needs of the gifted. As Reeve observes, "Many teachers will not know how to implement what they are requested to do in the conflicting reports that have been issued recently," or "In some of the (mathematics) reports, the people making them do not seem to know what they wish to do, or how it ought to be done. Many of the recommendations made could not be carried out and, in some cases, many teachers of mathematics, unless they already know what to do, would not benefit from anything that they read in some of these reports."[6]

What follows in the present article is a description of a plan the writer has tried for seven years. It is designed to meet the needs of the bright student. But more than that it is designed to meet some of the limitations of time imposed on the modern teacher. Its philosophy involves giving the bright child advanced mathematics rather than more difficult problems with the same mathematics context.

A plan for the bright child in the ninth year. Perhaps we should explain the reason for the emphasis on the ninth year. This observer suspects that too many of us wait until the junior or senior year for enriching the program for the fast learner. This seems natural. By process of elimination only the better students remain and the quality of instruction is perhaps given a natural lift. But is it wise to wait? A ninth grader possesses certain attributes of eagerness to learn that often are lost in the coming sophistication of adolescence. It would seem wiser to strike while the pupil is younger and more impressionable.

What then for this impressionable ninth grader? Let us assume an average class size of 25 to 35 (at present the writer has a class average of 33). In order to continue to instruct the whole class and still make provision for the bright child, it is well to set several well-defined goals:

1. Decide on a process of picking out the best students in the class.

2. Be prepared to present a specific list of assignments that will cover the usual ninth-grade content.

3. Be prepared to go beyond the ninth-grade content into what is called in New York State "Intermediate Algebra."

4. Be prepared to define very precisely what role the bright student is to play in class, what your method of checking work will be, what measurements he will be subjected to, and what time limitations he has for accomplishing the work set out.

[6] *Op. cit.*, p. 3.

Picking the bright student. This procedure has proved reliable and quick (we must remember the point about time). For five or six weeks observe the performance in usual classwork and examinations of those who seem to learn rapidly and who get high marks. Experience has shown that on the average a class of 30 will have 4 to 6 in this category. On the basis of other criteria such as work habits, performance in other subjects, and possibly IQ results obtained from the guidance office (ten minutes), you might increase or decrease this number by one or two. Set yourself the first week in November as a deadline for choosing those who are to go on an accelerated plan. You can write the list for each class while you are giving a test. Since most teachers can identify these select few anyway, this whole procedure will take less than an aggregate of 60 minutes over a spread of six weeks.

Presenting the program to the student. Plan, as many mathematics teachers do, to have the class spend the last half of a period doing the next day's assignment. At that time call up to your desk the four or six students you have selected. Broach the plan to them. All of us have our own methods of talking to children. For the writer, the following simple statements and questions have proved effective and time-conserving.

1. I see that you are doing well in here and that you seem to like mathematics.

2. I noticed that sometimes you understand something and have to sit there while I explain it many times to someone else who can't see it right away.

3. How would you like to work on ahead of the class? If you get through elementary algebra before the year is over, you'll be able to go into some things that you will have next year and the year after. You'll be more ready to do them by then.

As artificial as this motivation is, it seems to have great appeal to the ninth grader.

There will rarely be refusals. This year there were two out of 34 children approached who indicated they would rather stay with the group. Moreover, there were 5 people who were not asked, who, upon hearing of the plan, asked to join.

What is the program? To meet the specification of brevity the program must be kept simple. The following explanation is usually adequate:

1. Here is a list of assignments I have picked out of our text. (Dittoing and typing time—one hour).

2. You are to do these assignments. You are not to do the assignments given to the rest of the class.

3. Get a notebook that you can hand to me occasionally. Write the page and number of the problem and do all the problems in the notebook. I suggest that you paste the list of assignments on the inside cover of the notebook.

4. In class you may pay attention to what I am saying, or if you choose you may work on your list of assignments. You may volunteer answers for the work the class is doing or you may not.

5. I will expect you to be ahead of the class all the time. Go as fast as you wish. Do some at home every night. Since you will not have to do the regular assignment you will have time to do this.

6. Whenever you get "stuck" see me when I am free during class, or after school or at any other time I can help you. (This has borne the remarkable result of the bright pupil's seeking information about new work before class instruction is given.)

This requires about fifteen minutes. The usual question asked concerns the deadline for the problems on the first list. The only insistence made is that the student attempt to keep ahead of the class. With a total of two hours and fifteen minutes planning over a period of six weeks, we have a plan under way.

What mathematics should the plan cover? While some programs vary, generally the introductory algebra course covers: (a) general numbers and formulae, (b) the axioms for working with equations, (c)

positive and negative numbers, (d) elements of dependence and mathematical relationship, (e) introduction to the Cartesian Coordinate System, (f) solution of equations with two variables, (g) factoring, (h) solution of quadratics, (i) a discussion of fractions and techniques for handling denominators in equations, (j) a further development of the language of algebra through exponents, (k) some elementary aspects of geometric figures, (l) the trigonometry of the right triangle.

Our first obligation is to see that the bright student masters this required work. The *Fifteenth Yearbook* notes, "It is not desirable to extend study to new topics at the expense of thoroughness. Pupils with able minds often believe the quick perception that comes to them is complete understanding when it is not—able students should be guided into the habit of thoroughness, and dilettantism should not be encouraged."[7]

This plan has several safeguards against the above eventuality:

a. The student receives in class the regular instruction which, because it is aimed at a slower comprehension than his, insures his understanding of the ninth-grade content.

b. That he may listen or work on his advanced work allows him to utilize his time more efficiently in class, enabling him to complete many more problems. The results have shown that thoroughness is not neglected.

c. The continuous record of assignments makes for a desire to have that record a good one.

Having exhausted the elementary algebra text, the student is tested. In this connection several standardized tests have been used. A teacher-made test can fill the gaps in these instruments. Another good evaluation of the student's readiness to move on is obtained through personal interview. Since, in practice, each advanced student reaches this stage at different times, the teacher might reasonably have the twenty minutes or so necessary for this interview.

Advanced work in algebra. The usual departure for more advanced work is the Cartesian coordinate system. There are many excellent textbook series which provide for Algebra I and Algebra II. In most of these texts the second book usually reviews the graphing concepts covered in the first course text and then moves on into more complex graphing situations. After a review of this, usually brief, the student enjoys probing into circles, ellipses, hyperbolas, and parabolas. A great deal of the work here reinforces basic graphing concepts before they are forgotten. The parabola is explored quite extensively. Children find it fascinating to note the effect on the graphed figure if the linear part of the parabolic equation is negative or positive as in $y=x^2-2x+6$ to $y=x^2+2x+6$. There are other obvious avenues of challenge in the parabola. The solution of simple quadratics through graphing proves stimulating and leads to a deeper consideration of the quadratic function. The sum of roots, product of roots, and imaginary solutions are considered. The character of the roots of quadratics is easily grasped. We then go back to graphical solutions of simultaneous equations involving quadratics.

While they are graphing second-degree equations, about half the students on advanced work ask a standard question. Being used to the slope-intercept method of graphing linear function, the query inevitably occurs, "How can we tell what is the slope of a curved line at a point?" Here the purpose of certain activities of the calculus can be discussed. For the unusually bright, probably six out of a ninth-year class of 130, this has meaning.

The discussion of rate of change in graphing usually leads to more work on variation. In our text there is an excellent chapter on dependence and variation which the pupils will have covered. This work is now extended. More difficult

[7] *Op. cit.*, p. 146.

formulas are evaluated. For example: $F=mv^2/g$. What happens to F when V is doubled? There is a whole series of formulas with good practical applications which can be studied here. Formulas obtained from a physics text are especially studied.

The study of elementary arithmetic and geometric series follows formula work. This is always fascinating to the brighter freshmen. They are encouraged to work out their own formulas (short cuts, we call them) for finding the sum of an arithmetic series. Formal work follows. The sum of a convergent infinite series is always intriguing to the young mind. Somehow he thinks the sum should get larger and larger beyond the limit.

The above extensions of elementary algebra represent the program followed by the very bright who complete the first year work by March or April (in one case in February). Time alone can limit the extent to which these people can go. Logarithms are not beyond them. A further development of trigonometric ratios is occasionally tried.

Other programs for the advanced student. The program noted is not applicable to the student who finishes the first year work late in April or early May. For this person, a look at some of the elements of statistics is fruitful. In my classes several of the advanced students have enjoyed some ideas devoted to statistics.

From here, if time allows, the relation between the axioms of geometry and those of algebra are noted. A geometry text is perused. In this connection, some of the advanced students ask to go on into geometry rather than into further elements of algebra. Some of them do so.

What are the results of the program? The main question at this point is, "How does the whole thing work out?" It would be untrue to maintain that the results are always spectacular. Yet certain specific things appear to happen:

1. *Discovery*—"Insightful problem-solving is characterized by sudden discovery and high retentitiveness."[8] Professor Fehr is probably referring to some such incident as the time when one of the advanced students discovered he could solve many quadratics with one graph by letting y in the equation $y=x^2+2x+1$ vary and assume other values besides zero. By working on his own with proper guidance, the element of discovery is always present for the advanced student. In many cases the teacher doesn't spoil this adventure for him.

2. *Personal evaluation*—"The greatest growth in the development of independent work habits comes when the students evaluate and accept responsibility for their own progress."[9] Professor Kidd probably would approve of this program, if the present writer correctly interprets the quoted statement. There is ample opportunity for the advanced student to note his progress, to clean up his work habits, and to be responsible for completing a given amount of work at his own rate.

3. *Individual learning*—Let us borrow from Professor Kidd again. "Learning is an individual matter—the important idea is that the learner himself must react—he is the one who must be actively engaged in the learning process. There is no substitute in the classroom for the student's thinking for himself."[10]

4. *Content is broadened*—That the advanced student is able to acquire more mathematics is obvious. Two students working under this plan were able to achieve grades in the 90's on Intermediate Algebra examinations at the end of their freshman year. Other students gained great satisfaction from seeing how much of the Intermediate Algebra Regents examination they could do.

[8] Howard Fehr, "The Role of Insight in the Learning of Mathematics," THE MATHEMATICS TEACHER, XLVII (October 1954) p. 392.

[9] Kenneth P. Kidd, "Improving the Learning of Mathematics," THE MATHEMATICS TEACHER, XLVII (October 1954) p. 395.

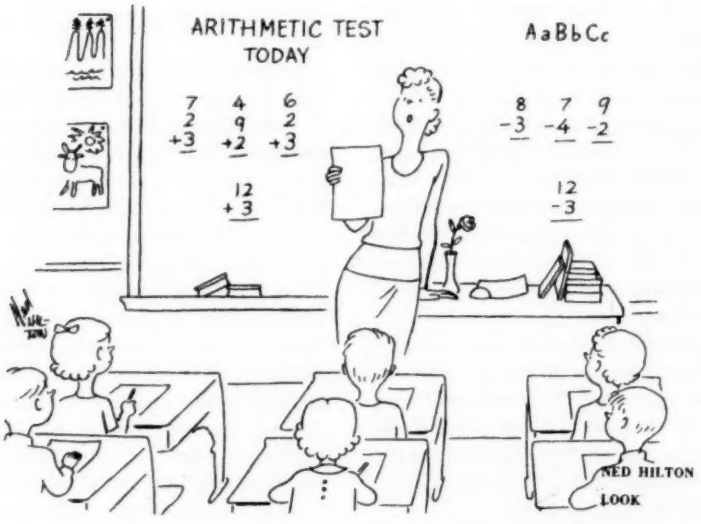
[10] *Ibid.*, p. 395.

5. *A coherent structure*—"Attention given to the main ideas of mathematics will help students to relate what would otherwise be isolated facts into a coherent structure."[11] At the end of the year, a half hour or so is spent with the advanced students discussing such major ideas as (1) of what value is the use of general numbers to the mathematician? (2) How does the equation enable us to handle difficult situations? (3) What is the connection between the number of unknowns in a situation and the number of relationships between the unknowns? (4) Why do techniques used for linear equations fail in the quadratic? (5) What more powerful techniques do we need to study the quadratic? (6) Why was the invention of the Coordinate System important? (7) Of what value are logarithms? Name a situation that logarithms facilitate for us that would be extremely difficult otherwise. (8) What is the value of statistical representation?

[11] *Ibid.*, p. 398.

Summary

It seems that we ought to take the problem of the bright student out of the category in which Mark Twain placed the weather. Besides talking, we can do something for the bright student. Despite teacher loads, there are ways. One way has been presented. Whether it is any good or not is hardly the point. At least more of us ought to try.



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"Entries will be judged on the basis of accuracy and neatness of presentation, and the decision of the teacher will be final."

• DEVICES FOR A MATHEMATICS CLASSROOM

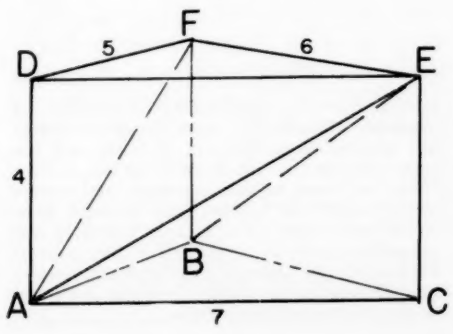
Edited by Emil J. Berger, Monroe High School, St. Paul, Minnesota. Anyone who has a learning aid which he would like to share with fellow teachers is invited to send this department a description and drawing for publication. If that seems too time-consuming, simply pack up the device and mail it. We will be glad to originate the necessary drawings and write an appropriate description. All devices submitted will be returned as soon as possible. Send all communications to Emil J. Berger, Monroe High School, St. Paul, Minnesota.

A paper model for solid geometry

by Ethel Saupe, Tracy, Minnesota

A theorem from solid geometry that is particularly difficult to visualize by means of drawings alone is the one which states that the volume of a triangular pyramid is one-third the product of the area of the base and the altitude. Appropriate three-dimensional models for use in developing the proof of this theorem are available commercially. However, home-made paper or tagboard models serve quite as well, and when a suitable set of patterns for construction is available, each student can have his own model. It is the purpose of this brief note to provide such a set of patterns.

Figure 1



Let the triangular right prism illustrated in Figure 1 be basic to the size and shape of the three equivalent pyramids that are needed. If sides AB, BC, and AC of the triangular base are respectively 5, 6, and 7 units, and altitude CE is 4 units, then $AE=\sqrt{65}$, $BE=\sqrt{52}$, and $AF=\sqrt{41}$ (units). We include the following approximations for the convenience of the reader:

$$AE=\sqrt{65}\approx 8.062\approx 8\frac{2}{32}$$

$$BE=\sqrt{52}\approx 7.211\approx 7\frac{7}{32}$$

$$AF=\sqrt{41}\approx 6.403\approx 6\frac{13}{32}.$$

Patterns for the three triangular pyramids can be drawn with ruler and compasses. Start with any triangular face, and on its edges construct the three adjoining faces that are needed to complete the tetrahedron (triangular pyramid). Thus, if we start with triangle ABC, we would first construct triangle ABC with sides of 5, 6, and 7 units as previously suggested, then triangle ABE on edge AB, triangle

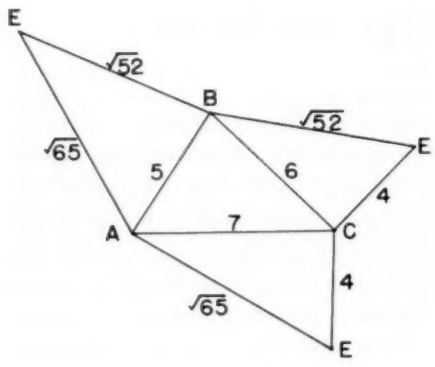


Figure 2

BCE on edge *BC*, and finally triangle *ACE* on edge *AC* (Fig. 2). The lengths of the edges *AE* and *BE* that are needed to complete the three last named triangles were tabulated above. If small flanges are added to one of each pair of the like-lettered edges, then tetrahedron *E-ABC* can be readily assembled with glue or paste. If the reader has Scotch-type pressure adhesive tape available, no flanges are needed.

In a similar manner draw and assemble pyramids *A-DFE* and *F-AEB*. The patterns needed to complete these solids are illustrated in Figures 3 and 4.

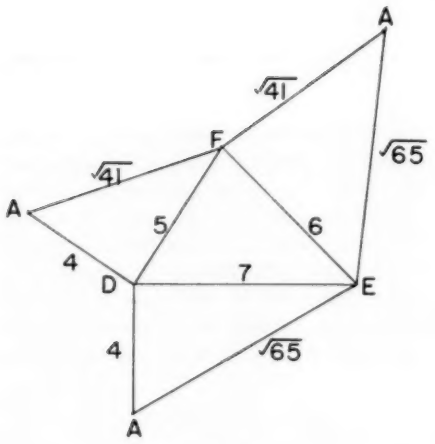


Figure 3

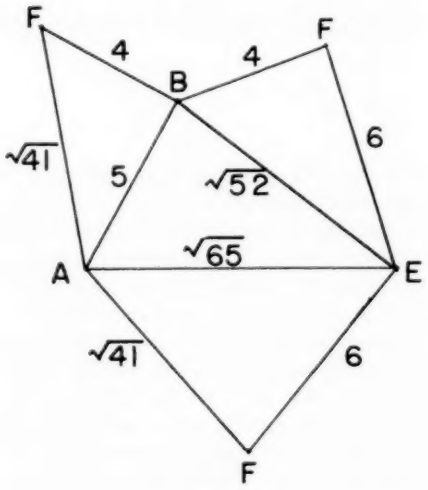


Figure 4

Master models in writing

When James Joyce faced the problem of showing the infinity of time, he solved it this way:

"You have often seen the sand on the seashore. How fine are its tiny grains! And how many of those tiny little grains go to make up the small handful which a child grasps in its play. Now imagine a mountain of that sand, a million miles high, reaching from the earth to the farthest heavens, and a million miles broad, and imagine such an enormous mass of countless particles of sand multiplied as often as there are leaves in the forest, drops of water in the mighty ocean, feathers on birds, and imagine that at the end of every million years a little bird came to that mountain and carried away in its beak a tiny grain of sand. How many eons upon eons of ages before it had carried away all. Yet at the end of that immense stretch of time not even one instant of eternity could be said to have ended."—From *A Portrait of the Artist as a Young Man.*

• HISTORICALLY SPEAKING,—

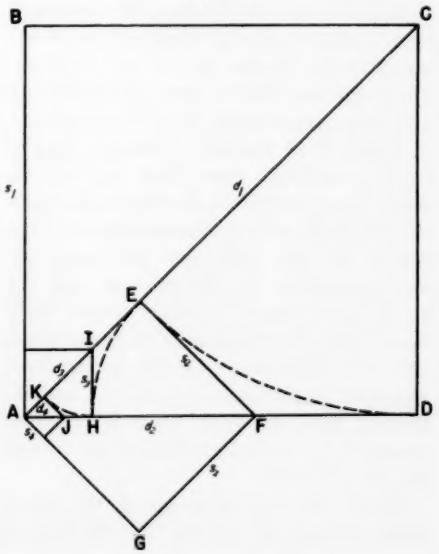
Edited by Phillip S. Jones, University of Michigan, Ann Arbor, Michigan

Irrationals or incommensurables II: the irrationality of $\sqrt{2}$ and approximations to it

In the preceding article on "Incommensurables"[1] the Pythagorean *proof* that the side and diagonal of a *square* are incommensurable using the number theoretic ideas of even and odd was summarized, and Kurt von Fritz' conjecture that the first incommensurables *discovered* were actually the side and diagonal of a *pentagon* was presented. Several persons[2] have also conjectured that the incommensurability of the side and diagonal of a square may also have been both *discovered* and *proved* by Pythagoreans using geometric diagrams and a repeated subtraction process similar to that described for the pentagon. In this note we present these processes in spite of the fact that there is no proof at all that the Pythagoreans proceeded in this manner. Our reasons for doing this are: (1) The procedures are interesting in themselves; (2) They lead to special cases of the continued fraction expansions, which in a sequel we will discuss as an approach to irrationals that actually developed hundreds of years later; (3) They can be related to geometric-algebraic theorems known to be Pythagorean in their origin; and, finally, (4) All of these aspects of this approach to irrationals have possible utility for the enrichment of teaching.

In the square $ABCD$ of Figure 1 draw the arc DE in order to lay the side $CD = S_1$ off on the diagonal $AC = d_1$. If now we construct a second square, $AEFG$, we have $AE = d_1 - s_1 = s_2 = EF$. Further, $EF = FD$ since both are tangents to the arc ED from external point F. Hence the diagonal d_2, of the second square is $AF = s_1 - s_2 = d_2$. Laying off $s_2 = AE = EF$ on AF, we determine AH, which we use as s_3, the side of a third square. In this third square we see that $s_3 = d_2 - s_2$ and $d_3 = AE$

Figure 1



[1] P. S. Jones, "Irrationals or Incommensurables I: Their Discovery," THE MATHEMATICS TEACHER, XLVIII (February 1956)

[2] Sir T. L. Heath, *The Thirteen Books of Euclid's Elements* (Oxford University Press, 1908), vol. I.

B. L. van der Waerden, *Science Awakening* (Groningen, Holland: P. Noordfhoff Ltd., 1954).

$-IE=s_2-s_3$. This process may be reiterated, forming smaller and smaller squares the sides and diagonals of which are given by the recurrence formulas

(1) $\qquad s_n=d_{n-1}-s_{n-1}$

(2) $\qquad d_n=s_{n-1}-s_n$

From formula (1) we see that if s_{n-1} and d_{n-1} have a common measure (a finite length which would be contained an integral number of times in each), this would also measure s_n. From (2) we see that if s_n and s_{n-1} have a common measure, this would also measure d_n. Now, going back to the beginning and looking at our iterative process, we see that if s_1 and d_1 have a common measure, this common measure must be contained an integral number of times in each of the sides and in each of the diagonals of each of our successively smaller squares.

The final conclusion that s_1 and d_1 are incommensurables now hinges on showing that s_n and d_n approach zero as a limit and that therefore the limit of the assumed common measure of s_1 and d_1 is zero or, in other words, the common measure does not exist. The Greeks did not have the limit concept in a modern form. In fact, if one insists that for them to have had the limit concept it must have been stated explicitly, then one could argue that the Greeks did not have it at all. However, they did in many places use the principle, stated explicitly as Proposition 1 of Book X of Euclid's *Elements*, that if from a quantity more than its half be taken away, and from the remainder more than its half, etc., the remainder will ultimately be less than any previously selected quantity. In the present case the previously selected quantity would have been the assumed non-zero common measure, and application of this subtraction principle would have lead to the contradiction that if there were a common measure, it would have had to have been less than itself.

From the vantage point of the twentieth century, we can see that the idea of a limit has been implicit in most approaches to the concept of an irrational, and in all *general* approaches from the earliest times to the present. Hence, in a sense, the history of the limit concept is a part of the history of the irrational. This limit concept, which is basic to the calculus too, has its roots in the early determinations of areas, volumes, lengths, tangents, rates, and sums of series by such persons as Democritus, Zeno, Eudoxus, and Archimedes with even some foreshadowings in earlier Babylonian and Egyptian work. Here, however, we will discuss the limit idea only as it is needed explicitly in dealing with the history of irrationals.

Plato (c. 400 B.C.) refers to "side" and "diagonal" numbers which seem to have been developed earlier by followers of Pythagoras, but for which some of the necessary theoretical justification is first found written out in Euclid's *Elements*. The ratios of associated pairs of these numbers, as we shall see, give the same successive rational approximations to the $\sqrt{2}$ which later grew out of the study of continued fractions and their limits.

In modern terms the side and diagonal numbers are defined by the recurrence relations

(3) $\qquad S_{k+1}=S_k+D_k$ and

(4) $\qquad D_{k+1}=S_k+S_{k+1}=2S_k+D_k.$

On the Pythagorean-philosophical basis that all numbers have their basis or root in unity, Plato, and probably the Pythagoreans before him, chose $S_1=1=D_1$. If we do this, we can quickly construct the table

k	S_k	D_k	D_k/S_k	
1	1	1	1/1 =1.	
2	2	3	3/2 =	1.5
3	5	7	7/5 =1.4	
4	12	17	17/12=	1.4167
5	29	41	41/29=1.4138	
6	70	99	99/70=	1.4142
:	:	:	:	

The entries in the last section of the preceding table have deliberately been arranged alternately in two columns. Those in the first column are less than the $\sqrt{2}$, and those in the second column are greater than $\sqrt{2}$, but both converge to the $\sqrt{2}$. This is typical of the successive convergents of continued fraction representations of irrational numbers. To derive a continued fraction representation of $\sqrt{2}$ we regard it as a fraction and rationalize its numerator as shown at the bottom of this page.

If now we compute the successive convergents of this continued fraction obtained by "chopping off" the fraction at the points indicated by the vertical dotted lines, we get exactly the fractions found in our table under D_k/S_k.

This same continued fraction could have been obtained geometrically by reasoning from Figure 1 as follows at right:[3]

$$\frac{d_1}{s_1}=\frac{AC}{AD}$$

$$=\frac{EC+AE}{AD}=\frac{AD+AE}{AD}$$

$$=1+\frac{AE}{AD}$$

$$=1+\frac{1}{\frac{AD}{AE}}=1+\frac{1}{\frac{s_1}{s_2}}$$

$$=1+\frac{1}{2+\frac{AH}{AE}}=1+\frac{1}{2+\frac{1}{\frac{s_2}{s_3}}}$$

$$=1+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{\frac{s_3}{s_4}}}}$$

$$=1+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+}}}\;\cdot\;\cdot\;\cdot$$

[3] A slightly different approach to this is to be found in the translation of A. M. Legendre's *Elements of Geometry*, which was so influential in mathematics instruction in the early days of this country. See P. S. Jones, "Continued Fractions and Rationalizing Numerators," THE MATHEMATICS TEACHER, XLVI (March 1953), 190 ff.; and "Early American Geometry," THE MATHEMATICS TEACHER, XXXVII (January 1944), 39.

$$\sqrt{2}=\frac{\sqrt{2}}{1}=1+\frac{\sqrt{2}-1}{1}$$

$$=1+\frac{1}{\frac{\sqrt{2}+1}{1}}$$

$$=1+\frac{1}{2+\frac{(\sqrt{2}-1)}{1}}$$

$$=1+\cfrac{1}{2+1+\cfrac{1}{2+1+\cfrac{1}{2+1+\cfrac{1}{2+1+\cfrac{1}{2+1+\cfrac{1}{2+}}}}}}\;\cdot\;\cdot\;\cdot$$

A proof that the ratio of the side and diagonal numbers S_{k+1}, D_{k+1} approximates $\sqrt{2}$ as a limit as k increases can be easily arrived at on the basis of two theorems, which are regarded as Pythagorean in origin, although the first is found today as Proposition 10 of Euclid's Book II, a book containing much geometric algebra. This Proposition states: *If a straight line be bisected and produced to any point, the square of the whole line thus produced and the square of the part of it thus produced are together double of the square of half the line bisected and of the square of the line made up of the half and the part produced.*

Both the meaning and the proof of this are most easily seen if it is translated into modern algebraic symbolism. If (Fig. 2) AB is the line, C its midpoint, and D the arbitrary point to which it is extended, and if we let $x=AC=CB$ and $y=BD$ the theorem then states

$$(5) \qquad (2x+y)^2+y^2=2x^2+2(x+y)^2.$$

When written in this form, the proof of the theorem becomes merely a demonstration that this is an algebraic identity by expanding and collecting terms.

Euclid drew the auxiliary lines shown with $CE=x$. $\angle AEG$ is then a right angle and, in modern notation, his proof was:

$$\overline{AE}^2=\overline{AC}^2+\overline{EC}^2=2\overline{AC}^2$$

$$\overline{EG}^2=\overline{EF}^2+\overline{FG}^2=2\overline{EF}^2=2\overline{CD}^2$$

$$\overline{AE}^2+\overline{EG}^2=2(\overline{AC}^2+\overline{CD}^2)=\overline{AG}^2$$

$$\overline{AG}^2=\overline{AD}^2+\overline{DG}^2=\overline{AD}^2+\overline{BD}^2$$

$$\therefore\ \overline{AD}^2+\overline{BD}^2=2(\overline{AC}^2+\overline{CD}^2).$$

The second theorem of Pythagorean origin to which we referred says that if S_{k+1} and D_{k+1} are derived by the recurrence formulas (3) and (4) from $S_1=1=D_1$ then

$$(6) \qquad D^2_{k+1}=2S^2_{k+1}\pm 1.$$

This can be easily verified for the entries in our table. Before we prove it however, let's see how it justifies our earlier

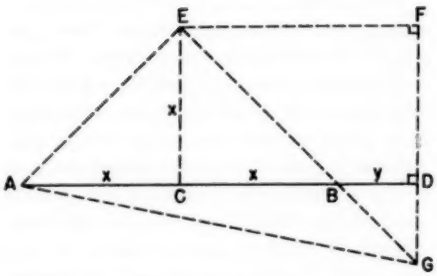


Figure 2

statement that $D_{k+1}/S_{k+1}\approx\sqrt{2}$. Dividing (6) by S^2_{k+1} and taking square roots we have

$$(7) \qquad \frac{D_{k+1}}{S_{k+1}}=\sqrt{2\pm\frac{1}{S^2_{k+1}}}.$$

From this we see that

$$\lim_{S_{k+1}\to\infty}\left(\frac{D_{k+1}}{S_{k+1}}\right)=\sqrt{2}.$$

Again, this application of the limit idea is modern, not Pythagorean.

To derive (6) we rewrite identity (5) in the form

$$(8) \qquad (2x+y)^2-2(x+y)^2=2x^2-y^2.$$

If we let $x=S_k$, $y=D_k$ then $(2x+y)=D_{k+1}$, and $(x+y)=S_{k+1}$ so that from (8) we derive

$$(9) \qquad \begin{aligned} D^2_{k+1}-2S^2_{k+1}&=2S^2_k-D^2_k \\ &=-(D^2_k-2S^2_k). \end{aligned}$$

If now we take $S_1=1=D_1$ we have $D^2_1-2S^2_1=-1$ and $D^2_{k+1}-2S^2_{k+1}=\mp 1$ as k is even or odd. Thus $D^2_{k+1}=2S^2_{k+1}\pm 1$ as in (6) with the plus sign applying when k is odd.

Exactly how and when these relationships were discovered and when they were proved is very hard to ascertain. Formula (7) and the limit process applied to it are ours. We have called the side and diagonal numbers S_{k+1} and D_{k+1} and the theorems represented by formulas (5) and (6) "Pythagorean," meaning that there is evi-

dence indicating that their source was not the man Pythagoras (c. 540 B.C.), but the school which he founded. This evidence lies largely in remarks attributed to Plato (c. 400 B.C.) by Proclus (A.D. 450) and in the works of Theon of Smyrna (c. A.D. 125) and Iamblichus (A.D. 325). Another "Pythagorean" theorem related to these side and diagonal numbers when stated in modern notation is

$$(10) \qquad \sum_{k=0}^{k=i-1} D_{k+1}^2 = 2 \sum_{k=0}^{k=i-1} S_{k+1}^2$$

when i is even. You can verify this easily for the numbers in our table.

One possible source of the idea of side and diagonal numbers may be observed in the geometry of the square with which we began this note. Our formulas (1) and (2) for the sides and diagonals of smaller squares combine to give

$$(11) \qquad d_n = 2s_{n-1} - d_{n-1}.$$

If we now conceive of constructing from a little square, with s_n and d_n as side and diagonal, larger squares by reversing the first construction we discussed, we can derive recurrence formulas for the sides and diagonals of the larger squares. To do this substitute $S_k = s_n$, $S_{k+1} = s_{n-1}$, $D_k = d_n$, $D_{k+1} = d_{n-1}$ in (1) and (11) and solve for S_{k+1} and D_{k+1}. This gives our side and diagonal formulas (3) and (4).

So far in our discussion of irrationals all of the methods and proofs have been *special*, chiefly related to the irrationality of the $\sqrt{2}$.[4] After the discovery of incommensurables by Hippasus and the work by early Pythagoreans which has just been related, the major contribution is that of Theodorus of Cyrene (c. 425 B.C.) who showed the square roots of the primes from 3 through 17 to be irrational. That these specific irrationals and only these were studied by Theodorus indicates that he was still using methods which did not generalize. On the other hand, Euclid (c. 300 B.C.) treats ratios in two different ways. At times he uses a definition which applies only to commensurable quantities and is Pythagorean in spirit. More often he uses a definition of ratio and equal ratios, which was formulated by Eudoxus (c. 375 B.C.) probably on the basis of earlier suggestions by Archytas (400 B.C.) following ideas attributable to Antiphon and Byson (c. 425 B.C.) The Eudoxian approach was a *general* approach to the ratio of incommensurables based on an elaborate definition of equal ratios and using as tools in applying the definition "the method of exhaustion," which is an extension of the subtraction process we have been discussing. These matters and their relationships to what we have called the axiom of Archimedes provided a *general* theory of incommensurables and or equal ratios, which we will discuss in ouf next note.

[4] Other methods for proving the irrationality of $\sqrt{2}$ are to be found in Howard Eves, "The Irrationality of $\sqrt{2}$," THE MATHEMATICS TEACHER, XXXVIII (November 1945), 317; and Wesley J. Lyda, "Concerning the Irrationality of $\sqrt{2}$," THE MATHEMATICS TEACHER, XXXIX (April 1946), 176.

The progress of mathematics!

"God ever geometrizes."—*Plato (430–349 B.C.)*
"God ever arithmetizes."—*C. G. J. Jacobi (1804–1851)*
"Man ever arithmetizes."—*J. W. R. Dedkind (1831–1916)*

• MATHEMATICAL MISCELLANEA

Edited by Adrian Struyk, Clifton High School, Clifton, New Jersey

Two notes on binomial coefficients

by Adrian Struyk

I. *In any binomial expansion the number of odd coefficients is a power of 2.*

This property of the binomial coefficients will be considered by using a diagram which will help us to "see" that the preceding statement is true.

Let us recall that the binomial coefficients are the elements of a triangular array known as Pascal's Triangle, sometimes called the Arithmetical Triangle or the Binomial Triangle. A small corner at the summit of this array is shown here. The numbers in any horizontal row are the

$$
\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 2 & & 1 & & & \\
 & & 1 & & 3 & & 3 & & 1 & & \\
 & 1 & & 4 & & 6 & & 4 & & 1 & \\
1 & & 5 & & 10 & & 10 & & 5 & & 1 \\
1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1 \\
1 & & 7 & & 21 & & 35 & & 35 & & 21 & & 7 & & 1
\end{array}
$$

terms of a power of the binomial $(1+1)$. The essential feature or *generating principle* of the array is that any number in a given row (except the end numbers, which are all 1's) is the sum of the two numbers nearest it in the row just above. Its basis is the identity

$$\binom{n}{r-1}+\binom{n}{r}=\binom{n+1}{r}.$$

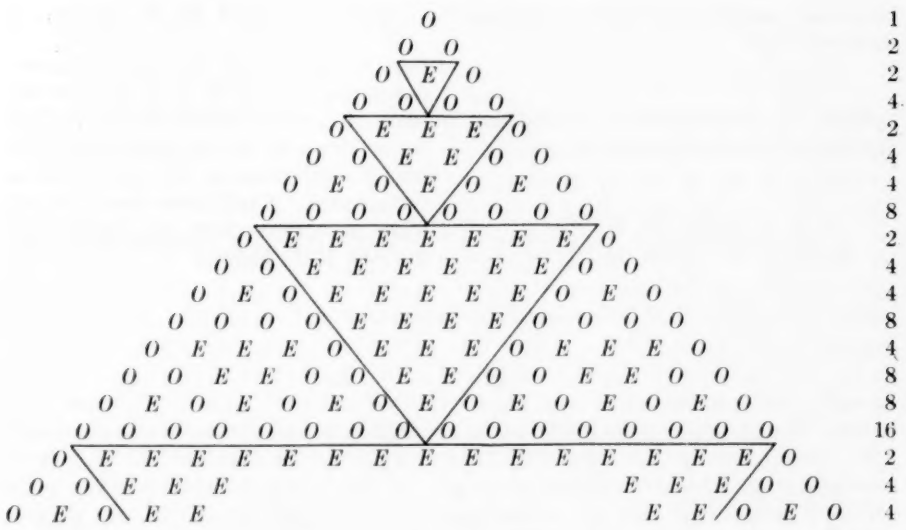
Now let us replace the numbers in the above triangle by symbols which take into account only divisibility by 2, writing E for each even number, O for each odd number. At the same time we must substitute for ordinary arithmetic the "addition facts" for odds and evens, viz.,

$$O+O=E, \qquad E+E=E,$$
$$O+E=O, \qquad E+O=O.$$

Operating with these in accordance with the generating principle of Pascal's Triangle, the new array can be extended as far as desired, the only further requirement being that O is to be supplied at both ends of each row. Thus we obtain the diagram on the following page.

Prominent in this diagram are the triangular sets of E's extending down the middle. Lines drawn just above these wedges of E's (including the single E in the third row) divide the array into trapezoidal sections. Each section in turn divides quite naturally into *three* triangles —the central inverted one, and two others rimmed by O's.

Choosing any one of the trapezoidal sections for scrutiny, let us pay particular attention to its two O-rimmed triangles. We note that each one reproduces exactly the whole triangular portion of the diagram lying above the section being considered. After reflecting a bit we realize that such repetition is the inevitable consequence of a row that contains only O's. For the E's produced in the next row act as a buffer between the O's supplied at the ends, so that, in effect, the diagram gets a fresh start. In the process of extending the diagram row by row, the same


O 1
O O 2
O E O 2
O O O O 4
O E E E O 2
O O E E O O 4
O E O E O E O 4
O O O O O O O O 8
O E E E E E E E E O 2
O O E E E E E E E E O O 4
O E O E E E E E E E O E O 4
O O O O E E E E E E O O O O 8
O E E E O E E E O E E E O 4
O O E E O O E E O O E E O O 8
O E O E O E O E O E O E O E O 8
O O O O O O O O O O O O O O O O 16
O E E E E E E E E E E E E E E E O 2
O O E E E E E E E E E E E E E O O 4
O E O E E E E E E E E E E E O E O 4

conditions that produced earlier rows are repeated *at both ends of each row.* Hence any one of the later rows of the array contains twice the number of *O*'s that appear in some particular previous row.

In the first and second rows of the diagram there are, respectively, a single *O* and two *O*'s—and 1, 2 are numbers that are powers of 2. Since generating the diagram after the first two rows involves successive doubling, no factor other than 2 is introduced in expressing the number of *O*'s in any subsequent row. It is evident, therefore, that our opening statement is true.

As corollaries we may note that the coefficients in the expansion of $(a+b)^n$ are

(1) alternately odd and even if $n=2^k-2$, $k=1, 2, 3, \cdots\cdots$;

(2) all odd if $n=2^k-1$;

(3) all even, except the first one and last one, if $n=2^k$.

In the column at the right of the diagram the entries list the number of *O*'s in the respective rows. The law of formation for this interesting sequence is obviously: The first term is 2^0; the first 2^n terms having been written, double these in succession to obtain the next 2^n terms.

II. *In an infinity of binomial expansions three consecutive coefficients are in arithmetical progression.*

It is not so much the fact itself, but rather an attendant phenomenon occurring in a table of values that we wish to present here.

We may suppose that

$$\binom{n}{r-1}, \quad \binom{n}{r}, \quad \binom{n}{r+1}$$

are in A.P., and write

$$2\binom{n}{r} = \binom{n}{r-1} + \binom{n}{r+1}.$$

In simplified form this relationship is expressed by the equation

$$(2r-n)^2 = n+2.$$

Now r and n are integers. Hence the preceding equation shows that n must be 2 less than a square. Therefore let

$$n = p^2-2, \qquad p \text{ an integer.}$$

Then

$$(2r-n)^2 = p^2,$$

and

$$2r-n = \pm p.$$

Since there are two values of r for a given n, let us write

$$r_1=\tfrac{1}{2}(n-p), \qquad r_2=\tfrac{1}{2}(n+p).$$

From the above relationships we derive the following endless sequences:

$p=2,\ 3,\ 4,\ 5,\ 6,\ 7,\ 8,\ \cdots$

$n=2,\ 7,\ 14,\ 23,\ 34,\ 47,\ 62,\ \cdots$

$r_1=0,\ 2,\ 5,\ 9,\ 14,\ 20,\ 27,\ \cdots$

$r_2=2,\ 5,\ 9,\ 14,\ 20,\ 27,\ 35,\ \cdots.$

Here we notice that the value of r_2 corresponding to a given n is equal to the value of r_1 corresponding to the succeeding n. Hence in the accompanying table, sets of coefficients in A.P. are found in juxtaposition. (Coefficients were selected, with permission, from an unpublished table by F. L. Miksa.)

r \ n	2	7	14	23	34	47
	1	1	1	1	1	1
1	2	7	14	23	34	47
2	1	21	91	253	561	1081
3	0	35	364	1771	5984	16215
4		35	1001	8855	46376	1 78365
5		21	2002	33649	2 78256	15 33939
6		7	3003	1 00947	13 44904	107 37573
7		1	3432	2 45157	53 79616	628 91499
8			3003	4 90314	181 56204	3144 57495
9			2002	8 17190	524 51256	13626 49145
10			1001	11 44066	1311 28140	51780 66751
11			364	13 52078	2860 97760	1 74171 33617
12			91	13 52078	5483 54040	5 22514 00851
13			14	11 44066	9279 83760	14 06768 48445
14			1	8 17190	13919 75640	34 16437 74795
15				4 90314	18559 67520	75 16163 04549
16				2 45157	22039 61430	150 32326 09098
17					23336 06220	274 11888 75414
18					22039 61430	456 86481 25690
19					18559 67520	697 31997 70790
20					13919 75640	976 24796 79106
21					9279 83760	1255 17595 87422
22					5483 54040	1483 38976 94226
23						1612 38018 41550
24						1612 38018 41550
25						1483 38976 94226

Edited by Lucien B. Kinney and Dan T. Dawson,
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A community approach to general mathematics

by Ethel Stubblefield, Consultant in Mathematics and Science,
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Fresno's Second Annual Universal Mathematics Tournament was held last April. Everyone—students and adults alike—felt that the occasion was profitable. Almost as soon as the awards were made, plans for staging next year's tournament were being discussed.

To provide the mathematical competence required for living in this complex world is the important function of general mathematics in the junior and senior high schools. Courses designed for this purpose must be practical and interesting. Their importance must be recognized not only by the teachers, but by parents and pupils as well. The Tournament has been an effective means for securing public cooperation in developing the kind of courses we need.

The math tourney is a part of a program started four years ago. At that time the mathematics teachers felt that something should be done to improve the arithmetic ability of many of the high school pupils. They wanted a course which would enable the pupils to cope successfully with the mathematical problems of everyday living, and at the same time have a status equal to that enjoyed by the more traditional courses.

A committee of three teachers and three businessmen spent the first year (1952–53) studying the problems and considering objectives and practical activities. Other businessmen and teachers were included at specific intervals. Twice during the year a large group of businessmen, Parent-Teacher presidents, and educators were drawn into the picture. The second year, the course was tried out in the classroom and revised in the light of that experience. In the spring of the following year, the first mathematics tournament was held. It was enthusiastically received, and all agreed it should be a yearly event.

Early in the fall of 1954, the Junior Chamber of Commerce was approached to be sure that it was planning to sponsor the affair again this year. In February a committee of teachers and representatives from the Junior Chamber of Commerce met at the home of the Chairman of the Universal Mathematics Committee to discuss plans. It was agreed that the Chamber would be responsible for two perpetual trophies to be given the winning junior and senior high schools. These men also agreed to take care of the publicity and furnish the judges. The teachers were responsible for the staging of the tournament. The date of the tourney was cleared by both groups, and a second meeting in March coordinated the plans.

The first tourney had been divided into three parts: mental calculations, calculations done at the blackboard, and word problems which were solved at the desks. It was decided to follow the same general plan this year. At the invitation of one of the new junior high schools, the tourney was held there. The all-purpose room was gayly decorated in math mobiles, and the

Parent-Teachers Association furnished punch and cookies during the intermission.

Each of the eight junior high schools was allowed two contestants, and each of the four senior high schools was allowed four contestants. Each school was allowed to select its representatives in any manner it chose. Actually, most of the schools did it on a strictly competitive basis, many of the try-outs being held after school hours. The teachers were furnished with copies of the questions asked in '54, in order to give them and their pupils some idea of the types of problems which would be given.

A label giving the team and number was pinned to the back of each student, and one judge was assigned four students. Each judge was provided with tally sheet for each part of the tournament. The first column gave the number of the problem, the second column the correct answer, and the next four columns were for the individual scores of the contestants he was to check. At the bottom of the column was a space to enter the number correct for each individual, and below this a space for the team total.

Each contestant was provided with a small chalkboard on which he wrote only his answer, displaying it at a given signal. At the end of each part, the team's total score was copied on the large scoreboard at the side of the room. The individual scores were never announced, but at the end of the tourney the top student in each division was awarded a small trophy. A sample of the score sheet used is given at the bottom of this page.

Part I consisted of twenty problems to be solved mentally. These calculations involved numbers only, including all three types of percentage computation. The majority of the operations stressed the use of denominate numbers typical in everyday transactions.

In Part II, which was done at large chalkboards, the junior-high contestants were given the first ten problems, after which they took their seats. The senior high contestants then went to the board and were given the second ten. These were all similar to those given in Part I except that the numbers were larger, more complicated, and more numerous. The allotted time, eight minutes for the oral problems and ten minutes for each group at the board, seemed to be a little longer than necessary.

During a brief intermission following Part II, the Sequoia Junior High Math Team put on a ten-minute demonstration which involved audience participation. The students used many short cuts and out-performed most of the adults. While this demonstration was under way, the scorekeepers finished figuring the total team scores for the first two parts of the tournament. Although the contestants remained in their seats, this also provided them with time for relaxation.

Immediately after the demonstration, a set of six word-problems was handed to each contestant. Two sets were used, one for the junior high students and one for the senior high. While contestants were working on Part III, the audience was given copies of both sets of problems, along with the answers, plus a bonus of six brain-teasers for any adults who needed extra work to keep them occupied.

At the end of fifteen minutes, the papers were collected from the pupils by pages, who delivered the proper papers to each judge. While contestants and audience

Team	Pupil		Pupil		Team
	No. right	Score	No. right	Score	Score
Mental Calculations	$x1$		$x1$		
Rapid Calculations	$x2$		$x2$		
Written Problems	$x3$		$x3$		
TOTAL SCORE					

went to the snack bar for refreshments, the judges and score-keepers totalled all the scores. When the scoring was finished, a gong brought everyone back to his seat. The President of the Junior Chamber of Commerce then presented each contestant with a Certificate of Merit, and the two top-scoring students with their trophies. After this, the two winning teams were presented the perpetual trophies. A suggestion that some comic trophy be presented to the member of the audience making the highest score was not carried out, but the audience did enjoy working the problems. There were many indications that the audience, both parent and student, left feeling that Universal Mathematics was certainly one of the more important subjects in the high school curriculum.

RESULTS OF MATH TOURNAMENT 1955

MENTAL CALCULATIONS	Number correct	
	Total	Your school
Actual number competing . . . 23		
1. Multiply 86 by 5.	20	
2. Add 28 and 6, subtract 4, divide by 5, multiply by 8. What is the answer?	22	
3. How many inches are in $2\frac{1}{4}$ yards?	8	
4. George gave the clerk a ten dollar bill to pay for a pair of shoes costing $8.19. How much change should he receive?	15	
5. 32 is what per cent of 40?	14	
6. What is the cost of 8 eggs if they sell for .54 a dozen?	8	
7. Multiply 67 by 73.	5	
8. What is $37\frac{1}{2}\%$ of 160?	15	
9. If the sales tax is 4 cents on each dollar, what would be the the tax on $8.50?	13	
10. How many cubic feet in a cord of wood which is 8 feet long, 4 feet wide, and 4 feet high?	10	
11. Find the perimeter of a rectangular table top that is 6 feet long and $2\frac{1}{2}$ feet wide.	13	
12. What is 120% of 45?	11	
13. A can of fruit juice weighs 3 lb. 2 oz. How much will 8 cans weigh?	17	
14. A sheet of tin is 20 inches wide. How many strips, $2\frac{1}{2}$ inches wide, can be cut from it?	12	
15. Multiply 125 by 64.	2	

RAPID CALCULATIONS	Total	Your school
Junior High		
Number competing . . . 15		
1. Addition: 2,978 4,632 5,182 469 6,285	11	

	Number correct	
	Total	Your school
2. Multiplication: 3,467 × 528	9	
3. Division: $67\overline{)15{,}477}$	12	
4. Which is larger: $\frac{5}{7}$ or $\frac{3}{4}$?	12	
5. Addition: $2\frac{3}{4} + 6\frac{2}{3} + 5\frac{1}{8}$	11	
6. Addition: 203.8 + 24.07 + 645 + .41	12	
7. Find the cost of 144 yards of bark cloth at .58⅓ per yd.	10	
Senior High		
Number competing . . . 8		
1. Multiplication: 7,364 × 258	6	
2. Division: $8.42\overline{)3304.85}$	5	
3. Find the total of the following commissions: \$356.75, 409.56, 432.84, 296.42, 514.68	8	
4. What was the average commission paid? Total was \$2010.25 for 5 commissions.	5	
5. \$91 is $16\frac{2}{3}\%$ of what amount?	5	
6. How many packages of frozen vegetables can be purchased for \$2.25 if each package costs $37\frac{1}{2}$¢?	5	
7. At 45 miles per hour how many feet will an automobile travel in 20 seconds (1 mile = 5,280 ft.)?	2	
8. What is the total amount of material represented by the following record: 43 yards, $17\frac{3}{8}$ yards, $29\frac{1}{4}$ yards, $15\frac{5}{8}$ yards, $30\frac{1}{4}$ yards, $18\frac{1}{3}$ yards	5	

WORD PROBLEMS	Number correct: Total	Number correct: Your school
Junior High		
Number competing . . . 15		
1. Using pi as $3\frac{1}{7}$, find the area of a circle whose diameter is 7 inches.	4	
2. The cash price of a refrigerator is \$120. The installment price is \$19.50 down and \$10 a month for 12 months. What is the installment price?	11	
3. What would be the interest on \$4400 for 75 days at $4\frac{1}{2}\%$?	8	
4. An aquarium is 22 inches long, 12 inches wide and 8 inches high. What is its volume in cubic inches?	10	
5. There are 18 boys in a class. This is 45% of the entire class. How many students are in the class?	12	
6. How many 6-ounce packages of pepper can be filled from a 30-pound box?	14	
Senior High		
Number competing . . . 8		
1. What is the interest on \$7500 at 6% for 42 days?	6	
2. At a recent election approximately 24,750 people voted. This was 55% of the registered voters. How many voters had registered?	4	
3. The tax rate in a city is $3\frac{1}{8}\%$. What are Mr. Smith's taxes on property worth \$24,000 and assessed at three fourths of of its value?	5	
4. How much gasoline will be saved by driving 2,450 miles in a car that averages 25 miles to a gallon instead of one that averages 14 miles to a gallon?	5	
5. How many dollars' worth of goods must an agent sell to earn \$160 if his commission is 4%?	8	
6. A television set that retails at \$200 is purchased on the installment plan. The terms are \$100 down and the balance in ten equal monthly payments. If the charge is 6% for interest and 2% additional for carrying charges, what is the amount of each of the ten payments?	5	

The teacher in a little backwoods school was at the blackboard explaining arithmetic problems, and was delighted to see that the gangling lad, her dullest pupil, was giving slackjawed attention.

Her happy thought was that, at last, he was beginning to understand. So when she had finished, she said to him, "You were so interested, Cicero, that I'm certain you want to ask more questions."

"Yes'm," drawled Cicero, "I got one to ask —where do them figures go when you rub 'em out?"

• MEMORABILIA MATHEMATICA

Edited by William L. Schaaf, Department of Education, Brooklyn College, Brooklyn, New York

The mysterious number PHI

by Sister Marie Stephen, O.P., Rosary College, River Forest, Illinois

The imagination is the storehouse of forms received through the senses, the reproductive and creative instrument of man's great desires and dreams; a little bit of body—a little bit of spirit. This quality makes the education of the imagination a vital necessity in the preservation of the integrity of youth.

The mathematician, particularly the geometer, depends on the imagination to provide the phantasms from which he abstracts his universal principles and then to serve again as a testing ground where these principles are resolved to their point of origin in the verification of mathematical judgements. Mathematics is the scientific enterprise of a creature that is on the confines between the spiritual and material orders.[1]

Mathematics needs the imagination as the loom on which it weaves its constructs, and the imagination needs the discipline and beauty of the order of mathematics that body and soul might converse together in a common language. It is an old story to those of us who have taught mathematics even on the college level that for the great majority today nine-tenths of the job is in selling the subject. One of the most convincing appeals is that made to the sense of wonder of the student. Young minds readily catch the flame of our contemplative thought and are quick to translate it into the world of their own creative abstraction, whether it be in art, or music, or science. This is the *image* of God at work in the rational creature.

A discussion between master and pupil in Plato's *Republic* points to the same truth:

Socrates: As I was saying, arithmetic has a very great and elevating effect, compelling the soul to reason about abstract number and rebelling against the introduction of visible and tangible objects into the argument.

Glaucon: That is true.

Socrates: Have you observed even the dull, if they have an arithmetical training, although they derive no other advantage from it, always become much quicker than they would otherwise have been?

Glaucon: Very true. Arithmetic is a kind of knowledge in which the best natures should be trained.

Socrates: Next we shall inquire into a kindred science.

Glaucon: You mean geometry?

Socrates: Exactly so.

Glaucon: Clearly you are concerned with that part of geometry which relates to war.

Socrates: Yes, but for that purpose a very little geometry or calculation will be enough. The question relates rather to the greater and more advanced part of geometry for *that* tends to make more easy the vision of the idea of the good and thither all things tend which compel the soul to turn her gaze towards that place where is the full perfection of being which she ought by all means to behold. Geometry compels us to view *being* and *becoming*. The knowledge at which geometry aims is knowledge of the eternal and

[1] Vincent Smith, *St. Thomas on the Object of Geometry* (Milwaukee: Marquette University Press, 1954), p. 53.

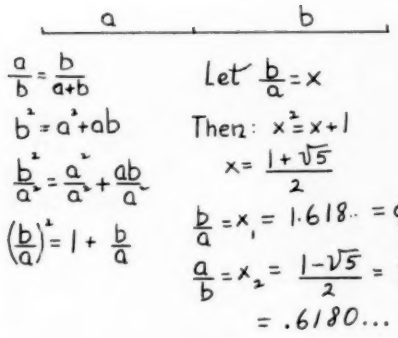

a
b
$\frac{a}{b} = \frac{b}{a+b}$
Let $\frac{b}{a} = x$
$b^2 = a^2 + ab$
Then: $x^2 = x + 1$
$\frac{b^2}{a^2} = \frac{a^2}{a^2} + \frac{ab}{a^2}$
$x = \frac{1+\sqrt{5}}{2}$
$\left(\frac{b}{a}\right)^2 = 1 + \frac{b}{a}$
$\frac{b}{a} = x_1 = 1.618\ldots = \phi$
$\frac{a}{b} = x_2 = \frac{1-\sqrt{5}}{2} = \frac{1}{\phi}$
$= .6180\ldots$

Figure 1

not anything passing or transient. It will draw the soul toward truth and create the spirit of philosophy and raise up that which is now unhappily allowed to fall down.

One of the most powerful illustrations of this quality of mathematics to open up new horizons to the imagination of man is found in the remarkable number *phi* (Φ) and its relation to life and art. The remainder of this paper will be devoted to this mysterious irrational, for so many centuries the fruitful source of new *ideas* to men of every profession. The Greeks first knew the number as that arising from the division of a line in extreme and mean ratio, i.e., so that the shorter section is to the longer as the longer section is to the whole line (Fig. 1).

Now we construct a rectangle of these dimensions: length $1+\sqrt{5}$ and width 2, as shown (Fig. 2). The *phi* rectangle, as it is called by mathematicians, possesses great dynamic beauty. If from the *phi* rectangle a square is removed, another *phi* rectangle remains; if this process is continued, and we join the vertices of these reciprocal rectangles, a spiral closely approximating the logarithmic spiral is generated. It is interesting to note the connection between the Fibonacci series and this gnomonic spiral (Fig. 3). In the early thirteenth century Leonardo of Pisa, nicknamed Fibonacci as he was the son (*filius*) of one Bonacci, a medieval merchant, discovered the series

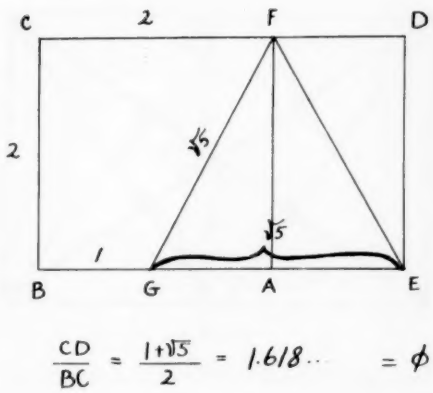


Figure 2

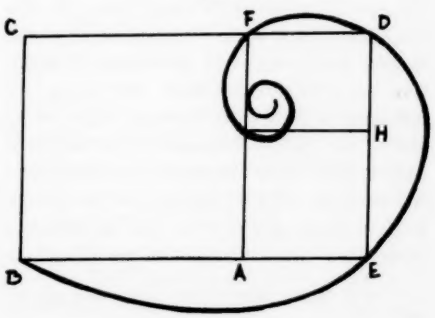
bearing his name while working on a problem involving the generation of rabbits. Many are no doubt familiar with it and its structure because of the considerable research done with this series in the theory of number:

1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144 · · · ·

The ratios of these numbers taken in either of these following orders approaches $.6180 = 1/\Phi$, which is the negative root of our quadratic quoted above. If we graph this series, we note its spiral is asymptotic to the *phi* spiral of the reciprocal rectangles.

In the sixteenth century, Luca Paciola, geometer and friend of the great Renaissance artists, rediscovered the "golden secret" and taught it to his friends. Leonardo da Vinci used it in laying out his

Figure 3



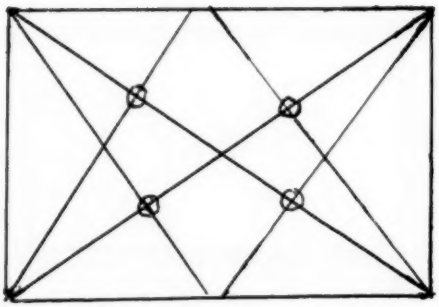


Figure 4

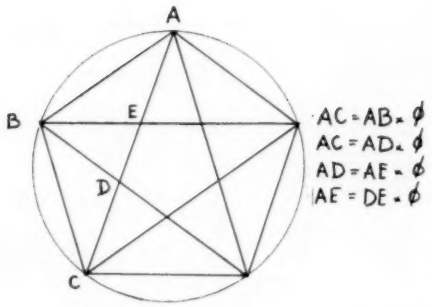


Figure 5

canvases in such a manner that the points of interest would be at the intersections of the diagonals and perpendicular from the vertices (Fig. 4).

Michelangelo painted his *Holy Family* in the pentagram whose long flank is to the side of the pentagon in which it is inscribed, in the *phi* ratio (Fig. 5). We find it again in Botticelli's *Magnificat* and in many other great masterpieces of that era (Fig. 6). The latest masterpiece done according to the golden rectangle is Salvador Dali's *Corpus Hipercubus* about which *Time* states:

> Dali originally entitled the work *Corpus hipercubus* (Hypercubic Body), explains that his painting is based on "the harmonious division of a specific golden rectangle."[2]

To trace its usage through the ages, go back to find it in the Great Pyramid of Cheops. The Egyptians used it in the triangular profile of this massive structure.

The Greeks attributed the discovery of the "golden number" to Pythagoras. Perhaps he became acquainted with it in the course of a mystical initiation of the Academy, for it seems this knowledge was taught under the seal of secrecy. Euclid, however, first calculated the ratio by dividing a line into extreme and mean ratio, and publicly taught this mode of calculation. The beautiful architectural refinements of the Greeks were accomplished by the use of what the Renaissance christened the "divine proportion," in everything from their most ordinary utensils to their wonderful temples. Professor Hammbridge of Yale has illustrated this copiously in his fine book on *Dynamic Symmetry*.

During that period of history which, in our ignorance, we often call the "Dark Ages," the knowledge of the golden number was transmitted by societies of artisans which one finds throughout Europe under the names: corporations, guilds, confraternities, workshops, lodges, etc. These men possessed all types of knowledge which was transmitted by word of mouth in the course of spiritual as well as technical initiations. From among them came the builders of the great medieval cathedrals. We see a notable instance of the *phi* composition in the royal doorway

Figure 6

[2] "Dali Makes Met," *Time*, LXV (January 24, 1955), 72.

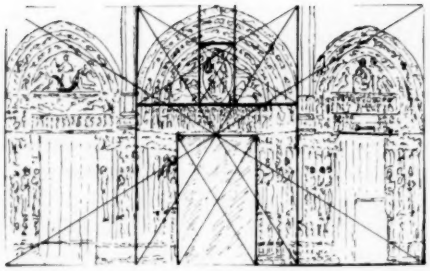


Figure 7

of the Cathedral of Chartres as shown in Figure 7.

At this point, since the expense of additional illustrations is prohibitive, the author would like to extend an invitation to anyone interested to use the many slides she has collected for demonstrations of this subject.

In the tower of Saint Jacques in Paris we find again as the architectural leitmotif, *phi*, in inverse progressions. At the corners, the buttresses rise in four superimposed layers, which diminish in size as they rise. The ratio thus established is exactly 1.618. The buttresses, like a human hand, whose proportions we shall see are the same, point toward the sky, while the three stories of windows which illumine the interior of the tower appear as a hand pointing down from the sky to the ground (Fig. 8).

In choosing such proportions the architect wished perhaps to express symbolically earth's forces rising to heaven, weakened and held back, and at the same time celestial light descending to the

Figure 8

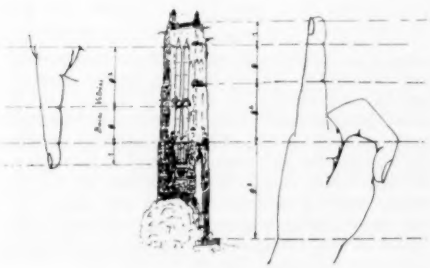


Figure 9

shadows below, which do not receive it except through man in whom these lower and higher forces are brought together. It is any wonder these great artists called the mathematical ratio establishing such proportions the "golden number"?

It was because of the prevalence of *phi* in nature that the artists copied it in their creations, and we therefore find them beautiful. The great botanist, Franci, acknowledges the ratio as a special algebraic invariant associated with the functions of life. If we examine the sunflower disc closely, we see the arrangement of its rhomboidal florets in counter logarithmic spirals of thirty-four and fifty-five, eighty-nine, and one hundred and forty-four. At Oxford one has been raised with the latter seedings.

Both the buds on a branch and the petals of a flower are so arranged that one petal lies directly above another only after one passes three, five or eight others in spiraling around the branch (Fig. 9). We are used to seeing the pentagonal design in our garden flowers, but look for it also in the pine cone and the veins of certain leaves. The spiral of the golden ratio appears also under the sea in the chambered nautilus and in many other shellfish. When we look to the heavens, we see it in the giant spirals of the nebulae.

Nor did God neglect the divine proportion in His masterpiece: man, who is a little cosmos joining all creation to its God.

In the temple of man's body, God locked the blueprint of creation, as He locked within his soul the Living Image of the Trinity. The human body, with legs and arms outstretched, may be encased in a pentagram. In the human hand, the length of the second digit multiplied by 1.618 gives the length of the first digit. The length of these two bones gives the length of the phalangine. The phalangine and phalangette together give the finger. The length of the hand up to the wrist joint multiplied by 1.618 gives the length of the forearm.

When a man stands erect, the distance from the ground to his navel, multiplied by 1.618, gives his total height.

The perfect human face may be analyzed harmonically into many golden rectangles. Dr. Christian Jacob of Buenos Aires has discovered the interesting proportion in the human brain.

Even our blood carries the "golden imprint," for it is capable of imposing in a supersaturated solution in the process of crystallization, a form in which a center appears. If the man from whom the blood has been taken has been in good health, the center can be located at a point on the diameter which is .618 of the whole width.

The golden number is not only, as it were, the "number of our physical body," it penetrates our very soul. Each time we discover the golden number in a work, natural or artificial, we recognize it. Some day cover your blackboard with rectangles of every dimension and witness your class's choice of the golden rectangle as that most aesthetically pleasing to it.

Beauty consists in a certain splendor and due proportion. By this single example of the "golden number and her daughters strong by the laws of heaven on whom rests and goes to sleep a honey-colored God,"[3] we can see the power of mathematics to move the human imagination, and the power of that imagination to design man's rhythms, orders, and harmonies according to these faint symbols of eternal Beauty.

Sir Edmond Whittaker, in an address given in his last years on earth, wrote:

> The principle that matter exists not for its own sake, but in order to help us in bridging the gulf that separates us from the divine, may be expressed in theological language by saying that nature has a sacramental quality; a principle that has long been recognized in religion and can now be admitted to be *not* alien to the philosophy of science.[4]

[3] "Le Nombre d'Or," *Plaisir de France*, Vol. I, December 1953, p. 27.

[4] Sir Edmond Whittaker, in an unpublished address given at the meeting of the *Royal Academy of Science*, London, England, 1952.

"Yes, science is a powerful tool, and it has an impressive record. But the humble and wise scientist does not expect or hope that science can do everything. He remembers that science teaches respect for special competence, and he does not believe that every social, economic, or political emergency would be automatically dissolved if 'the scientists' were only put into control. He does not—with a few aberrant exceptions—expect science to furnish a code of morals, or a basis for esthetics. He does not expect science to furnish the yardstick for measuring, nor the motor for controlling, man's love of beauty and truth, his sense of value, or his convictions of faith. There are rich and essential parts of human life which are alogical, which are immaterial and non-quantitative in character, and which cannot be seen under the microscope, weighed with the balance, nor caught by the most sensitive microphone."—*Taken from* "Science and Complexity," *by Warren Weaver*, American Scientist, *Vol 36, No. 4.*

A column of unofficial comment

National Council news

by Marie S. Wilcox, President, The National Council of Teachers of Mathematics

Readers of THE MATHEMATICS TEACHER will be happy to learn that the Board of Directors has appointed Henry Van Engen for a second term as editor of this magazine. This term will extend from June 1, 1956 until June 1, 1959. According to the constitution, editors for each of the Council journals are appointed for a period of three years. An editor may be reappointed but may not serve for more than two terms. All editors serve without compensation, but have allowances for editorial and secretarial assistance, and a limited allowance for travel expense.

Other news about Council publications includes the announcement that Foster Grossnickle has accepted the editorship of a new yearbook which will be concerned with the teaching of arithmetic. The planning committee to work with Mr. Grossnickle includes Ida Mae Heard, Irene Sauble, Dan T. Dawson, and C. L. Thiele.

The Committee on Supplementary Publications announces that two pamphlets on the education of the gifted child are in preparation, and that one concerning the education of the slow learner in mathematics will be started soon. E. P. Vance will edit one booklet concerned with the education of the gifted child. Howard F. Fehr is to be the author of the second booklet which will be part of the "How To" series. Elizabeth Roudebush has accepted the editorship of the pamphlet on the education of the slow learner in mathematics.

It was with deep regret that the Board of Directors accepted the resignation of Harold Larsen, who has served as editor of the *Mathematics Student Journal* since its inception. The success which the journal has enjoyed in these first years should be a source of satisfaction to Dr. Larsen and the associate editors who have worked with him during that time. The Board has secured the services of Max Beberman to serve the unexpired term of Dr. Larsen.

A new publication authorized by the Board at its recent meeting is a directory of members of the Council. Present plans are to publish a directory every two years with the first edition being made available this year. I feel sure that many people will consider this directory a "Who's Who" of professionally-minded mathematics teachers.

In the November issue, this column suggested that members of the Council consider the matter of the number of meetings which the National Council should sponsor each year. One suggestion which had reached the Board was that the Christmas meeting be discontinued. In considering this it should be noted that the past two Christmas meetings have been better attended than any previous Christmas meetings of the Council. In Washington this past December the registration was 907. This included some visitors, but each of these people probably participated in some part of the program arranged for that meeting.

The Council continues its policy to cooperate with other professional organizations. A guidance pamphlet which will be widely distributed will be written by one representative of the Council and one from the Mathematical Association of America. Irvin H. Brune will represent the Council, and A. L. Putnam, the Association.

The Council has for many years had representation on the Cooperative Committee of the American Association for the Advancement of Science. Our Board of Directors has now taken steps to become officially affiliated with this organization. Other leading mathematics organizations already have such affiliation.

Representatives of the Council attended the 1956 Regional Conferences on Teacher Education and Professional Standards in January. Our representatives and the Conference cities included: Jackson Adkins, Boston, Massachusetts; Ethel Grubbs, Washington, D. C.; Bess Patton, Atlanta, Georgia; George Hawkins, Chicago, Illinois; Gilbert Ulmer, Kansas City, Missouri; Eva Crangle, Salt Lake City, Utah; and Harold Bacon, San Francisco, California.

Have you read?

A.A.A.S. Cooperative Committee on the Teaching of Science and Mathematics. "Improving Science Teaching," *The Scientific Monthly*, Vol. 81, September 1955, pp. 109–114.

It is valid to assume you are all interested in improving teaching, which in turn will lead more people into science and mathematics, and ultimately will lead to better teachers and surer progress in science.

This article is a report to you from the Committee about a program to help in such improvement. Carnegie Corporation gave $300,000 to begin the study. You may be asked to help, because we all have responsibilities for the future. The article summarizes briefly where we are and what might be done. Some suggestions are: to consider what constitutes the proper kind and amount of training; how to encourage the research mathematician and the mathematics teacher at the same time; how to analyze curriculum offerings on college level, especially during summer sessions; how to obtain cooperation between state certification, high school teachers, education departments, and science or mathematics departments. Many other suggestions are made dealing with emergency measures, recruitment, salary, awards, consultant services, and others. You will want to be informed about this project, and also to call it to the attention of other interested people.

Engel, Leonard. "What Einstein Was Up To," *Harper's*, December 1955, pp. 69–74.

We should not excuse our ignorance of what Einstein was up to by passing off the dubious remark: Only a handful of men can understand Einstein's theories. Most mathematics teachers and many mathematics students with leanings toward physics, can gain a lot from this article. It gives a good picture of the scientific method in action, especially the role of testing old assumptions, and choosing new ones in the light of new data. The role of mathematics in scientific thought is clearly shown in its use as a language to describe natural phenomena, in showing relationships between theories, and in predicting previously unobserved phenomena.

Knowing, in a general way at least, the great scope of Einstein's achievements, you cannot help but be impressed again by his ability (and by the significance of mathematics) when you read that "he worked with the simplest of tools —pencil and paper."—Lyman C. Peck, *Iowa State Teachers College, Cedar Falls, Iowa*

Gant, P. "When Is a Parallelogram Not a Parallelogram?" *Mathematical Gazette*, XXXIX, September 1955, pp. 191–95.

This is just what your student in geometry needs to activate his mind in relation to the fundamentals of geometry. The article is composed entirely of several proofs which grew out of the question: Is a quadrilateral in which one pair of opposite sides are equal and one pair of opposite angles equal necessarily a parallelogram? These solutions are simple enough for most of your geometry students to understand. It will make them analyze proof more thoroughly. By all means, have them study this short discussion.—Philip Peak, *Indiana University, Bloomington, Indiana*

Reviews and evaluations

Edited by Richard D. Crumley, University of South Carolina, Columbia, South Carolina and Roderick C. McLennan, Arlington Heights High School, Arlington Heights, Illinois

BOOKS

Non-Euclidean Geometry, A Critical and Historical Study of Its Developments, Roberto Bonola, translated by H. S. Carslaw, ix+268 pp.; *The Science of Absolute Space,* John Bolyai, xxx+71 pp.; *Geometrical Researches on the Theory of Parallels,* Nicholas Lobachevski, translated by George Bruce Halsted, 50 pp., Dover Publications, Inc., New York, N. Y., 1955. Paper, $1.90; cloth, $3.95.

Three mathematical classics, each a valuable treatise for teachers of secondary-school mathematics, are included in one volume. This book can be recommended with enthusiasm for the personal library of the teacher and for the high-school library. Students with genuine curiosity about geometric ideas and non-Euclidean geometry can read most of this volume with interest and real profit.

The original Bonola manuscript is dated 1906, whereas the translator's preface is dated 1911. This work was based in part on the original manuscripts, translations of which are included in this volume, by John Bolyai and Nicholas Lobachevski, which were first published approximately 125 years ago. In comparison with the content and methods of traditional secondary-school mathematics, these original works could be classed as modern mathematics.

The importance of the translations from a historical point of view is emphasized in the introduction to the Bonola text by the eminent Italian mathematician Federigo Enriques, who writes: "It seems to me that this account, although concerned with a particular field only, might well serve as a model for a history of science, in respect of its accuracy and its breadth of information, and, above all, the sound philosophic spirit that permeates it."

An extensive translator's preface to the Bolyai text gives an excellent summary of the development of non-Euclidean geometry, and includes interesting biographic data on the Bolyais, father and son. Illustrative of detail, references are made to the originality in private life, as well as mathematics, of the father and to the fact that his wife, mother of John, was a beautiful, fascinating woman. The translator's introduction to the Lobachevski text is no less enlightening.

Appendix III of the Bolyai text, entitled "Light From Non-Euclidean Spaces for the Teacher of Elementary Geometry," is an essay by the translator, G. B. Halsted, which might well be required reading of all prospective mathematics teachers. This entire volume provides a ready reference by which those in teacher education can give more attention to non-Euclidean geometry and to the history of this exceedingly important topic.

Halsted refers to "The Science Absolute of Space," by John Bolyai, first published in 1832 as an appendix to a much longer book by his father, as "the most extraordinary two dozen pages in the whole history of thought" and "this wonderful production of pure genius, this strange Hungarian flower." While not all will agree with the place assigned by the translator to the "24 pages" in the history of thought, none would question that these pages are of extraordinary importance in the development of mathematics and a significant example of the "flower" of genius. For teachers who wish for themselves and for their students an opportunity to read translations of relatively recent mathematical papers of relevance to secondary-school mathematics, and of great importance, both the Bolyai and Lobachevski papers should be among first choices.—*John Mayor, University of Wisconsin, Madison, Wisconsin.*

Trigonometry, Roy Dubisch, The Ronald Press Co., New York, 1955. Cloth, xiv+396 pp., $5.00.

Modern Trigonometry, John C. Brixey and Richard V. Andree, Henry Holt and Co., New York, 1955. Cloth, xii+209 pp., $3.35.

Plane Trigonometry, C. R. Wylie, Jr., McGraw-Hill Book Co., Inc., New York, 1955. Cloth, viii+381 pp., $4.00.

Although there are many differences in approach to the subject matter of trigonometry in these three books, still, someplace in each a reader will find all the usual topics and formulas. That "someplace" may be the appendix in one book or the body of a chapter in another, but it exists for each. To this extent the authors have arrived at a compromise between their ideal courses, their points of view as stated in the prefaces, and the realistic situation faced by a hard-working author who wishes to write a book that teachers will buy.

It is interesting to note that the three books take for their emphasis the analytic aspects of

trigonometry, pointing out that it is these that are important in applications in mathematics, physics, and engineering. The third book staunchly defends the point of view that it is " . . . undesirable, if not actually impossible, to follow [the] exaggerated, completely theoretical approach in an elementary course" of a development of trigonometry in terms of functions of real numbers. *Modern Trigonometry* makes the transition from functions of angles to functions of real numbers in Chapter 5. However, the presentation seems such that a student may continue with the impression that the "true" trigonometry deals with angles, that real numbers are introduced as an interesting, albeit important, application. It is the first book, *Trigonometry*, that offers a consistent development of trigonometry as functions of real numbers, beginning with Chapter 1 and continuing to Chapter 8, where a transition to functions of angles occurs.

Hence, we can safely say that each of the books has a philosophy or "frame of reference." Too, the three books are careful to develop concepts slowly, with much attention given to details of derivations, to restrictions on the ranges of variables, and so on. The books contain many exercises that seem to be well chosen, but more will be said of this below as each book is considered in more detail. An experienced high school teacher may be a bit offended by one attitude common to the four authors; this is that a student studying from their textbooks knows no mathematics at all. Students are continually referred to as "beginning" their study of mathematics.

Trigonometry represents a school among authors of college textbooks whose members believe that undergraduate mathematics courses should be more closely related to the thinking in contemporary mathematics than is usually the case. Chapter 1 carefully develops the concepts concerning real numbers and analytic geometry that will be needed later. In this chapter appears a definition of *function* that identifies a function with the rule associating the members of two sets. This is not the definition that some mathematicians would choose; however, the choice must have been arrived at after much experimentation in a classroom and after much thought. It should be said, in addition, that the book is the most consistent of any of the books under review in the use of the concepts developed. Chapter 2 introduces the "arc length function (ALF)," and it is from this function that the whole development of trigonometry issues.

An effort is made to give an explanation and a setting for an exercise illustrating an application of a trigonometric concept. In Chapter 8, as was mentioned earlier, the transition from real numbers to angles is made. The functions of angles are treated in a matter-of-fact way as just another application of functions of real numbers. The chapter on the solution of triangles takes up only the law of sines and the law of cosines, although other formulas for the solution of triangles are given in an appendix. The logarithm function is defined as the inverse function of the exponential function, and the necessary restrictions are carefully explained. Complex numbers are introduced as ordered pairs of numbers $(r, \bar{\theta})$ where $r \geq 0$ and $0 \leq \bar{\theta} < 360$.

Modern Trigonometry leaves much to be desired in both its mathematical and pedagogical development. The authors make a great effort to include a large number of exercises indicating applications of trigonometry in physics, engineering, and calculus. However, very little is done to explain what the problem means; proper units are given and terminology is correct, but a student will find it difficult to draw meaning or understanding from these. As a result, most of the applications are "window dressing" and will be quite useless for motivation purposes.

On page 5 a character, one Calvin Butterball, appears in an exercise. Evidently he works on a farm, for he makes his first appearance as he hoes onions. From page 9, one infers he is fat, because he gets paid for losing weight in addition to hoeing onions. Page 16 tells us that Calvin has a girl-friend, Phoebe Small, with whom he goes on a date. Mr. Butterball has been promoted; he now packs onions in "a nice air-conditioned onion-packing plant." But on this job, Calvin turns into a dreamer, page 22, and ponders the problem of language. By page 40 he is sitting in his dentist's chair observing angles and trying to find the height of a flagpole. Page 75 brings the saga of Calvin to an end. It is not clear whether trigonometry proved too difficult for him or he drowned in the bay he had to cross to visit Phoebe. The authors are not at a loss; page 76 introduces a dull fellow, John Garfinkle, who has trouble with "signs." He, too, disappears after two more appearances in a similar predicament on pages 104 and 109.

Is this sort of thing necessary?

The roots of a quadratic equation should be stated as "$x = 4$ *or* $x = -1$," not as "$x = 4$ *and* $x = -1$"; this error appears first on page 11 and is repeated wherever the situation arises subsequently. On page 10 is a statement concerning equivalent equations from which it will be difficult for a teacher to draw a correctly stated definition for students. The definition of a^n on page 42 is curious. The reviewer has never seen the definition given in that particular form and wonders what the advantages are. The discussion of proof of identities seems incomplete without at least one example of what to do in case the steps are not reversible.

Plane Trigonometry starts on page 2 with an acceptable definition of *variable*, has a little trouble describing *absolute constant* and *arbitrary constant*, and gives essentially Dirichlet's definition of a *function*. Very quickly, after these preliminaries, the author gets to the main business of defining the trigonometric functions of an angle. His discussion of tan 90° is very good, but it would have been better if the table

on page 31 did not show "∞" in the column headed "tan θ" and opposite 90°. The author explains on the following page that such writing is merely a convention. However, experience with both students and teachers leads one to believe there is much confusion in the minds of both groups over this convention of writing "tan $90° = \infty$"; no amount of explanation or number of words of caution seems to remove the danger of misunderstanding. For that reason it seems wiser to avoid the convention.

In Chapter 5, page 115, where the author derives the functions of composite angles by use of the distance formula, appears the curious phrase "general point." How does a *general point* differ from an *ordinary point* or a *special point?* There has been a rash of invention of new terminology in recently published textbooks. It is doubtful if the new terminology helps understanding; perhaps a better approach would be to be careful in the usage of the words mathematics already has. Other examples of the author's attempts to make the language of mathematics easy for students are "general oblique triangle" on page 195 and "pure real number" on page 221.

On the whole, *Plane Trigonometry* represents a workmanlike, careful presentation of the materials of trigonometry that an instructor expects to find in a textbook with this title. The treatment, so far as students are concerned, is more detailed than usual, and more attention than usual is paid to developing concepts. Probably *Modern Trigonometry* will appeal to some instructors. Some will have a distaste for it because of its arch approach to exercises, its errors in presentation—not too serious, perhaps—and its out-of-setting-and-context problems of application. *Trigonometry* covers essentially the same topics as the other two books, but in an unexpected way. The author is most consistent of the three in his mathematics and his development. Students studying trigonometry from this book ought to end the course with an excellent idea of a connected mathematical development and some knowledge of the place of the materials of the course in applications.—*Myron F. Rosskopf, Teachers College, Columbia University.*

FILMS

Understanding Numbers: The Earliest Numbers (*IS-233*), *Base and Place* (*IS-234*), *Big Numbers* (*IS-235*), *Fundamental Operations* (*IS-236*), *Short Cuts* (*IS-237*), *Fractions* (*IS-238*), and *New Numbers* (*IS-239*). Audio-Visual Center, Indiana University, Bloomington, Indiana. 16 mm. films, 30 min. each, B&W; each film features Phillip S. Jones; purchase price of each film, $100; rental fee for each film, $3.50.

Produced originally for the Educational Television and Radio Center of Ann Arbor, Michigan, this series of seven films attempts to summarize and clarify some of the basic concepts pertaining to numbers. In doing this, much use is made of historical facts in the development of numbers. The setting for each of the seven programs is a lecture by Phillip S. Jones with a generous use of models, charts, and chalkboard.

In Program 1, *The Earliest Numbers*, man's first efforts to symbolize numbers are shown and described. Early contributions are pointed out which were related to the essential elements of a modern system of symbolization: base, place, symbols, zero, and decimal point. Early devices for computing with numbers are also shown.

In Program 2, *Base and Place*, the concepts and principles of base and place for a system of numeration are developed by a consideration of the binary system. The history of the binary system of notation is given, followed by the applications of this system to electronic digital computers and keysort cards used in business. The use of 10 as a base for our system of numeration is then emphasized along with our system of place-value.

Program 3, *Big Numbers*, deals with the ways in which scientists and mathematicians write and use very large and very small numbers. Exponents and scientific notation for symbolizing large numbers are explained and demonstrated, followed by a similar treatment for small numbers. Other uses of exponents shown are: compound interest, formula for the healing time of a wound, formula for belt friction, and a formula for assessing the value of a fleet of trucks. "Perfect numbers" are explained, and Euclid's formula for finding such numbers is given and demonstrated.

In Program 4, *Fundamental Operations*, modular arithmetics (modulo 5 and modulo 6) are explained and examined to illustrate the concepts of closure, commutative operation, and identity element. The associative property for addition and the distributive property of multiplication with respect to addition are illustrated, using our rational number system. The addition and multiplication tables for the modulo 5 system and the modulo 6 system are used to demonstrate that addition and multiplication are the two fundamental operations and that subtraction and division are inverse operations. Attention is now turned to our ordinary (rational) arithmetic. Explanation is given showing how a 10×10 table for addition, a 10×10 table for multiplication, and the principles of base and place can all be used to perform the two fundamental operations.

Program 5, *Short Cuts*, introduces logarithms by describing their historical development. Finger multiplication, lightning or cross-method of multiplication, and Napier's "bones" are each demonstrated. A simplified log table is introduced by showing powers of 10, and a larger but limited log table is used to illustrate the use of logarithms in computing. The historical development of slide rules is presented next, showing their dependence upon logarithms. The use of

logarithms in formulas in the areas of electricity and chemistry is also shown.

Program 6, *Fractions*, presents the derivation of the words "fraction," "numerator," and "denominator." The principles needed to add fractions are explained and demonstrated. Five meanings of a fraction are then described. The history of ways of representing fractions is traced, leading up to our present notation. The properties of decimal and duodecimal fractions are compared, and some of the arguments advanced by advocates of the duodecimal system are given.

In the final program, *New Numbers*, the history, characteristics, and uses of negative, irrational, transfinite, and complex numbers are described.

This series of films will be found to be very stimulating for anyone already interested in mathematics. For the uninitiated person, however, these films may only strengthen his belief that mathematics is mystical and full of magic. The tempo of presentation of ideas is so fast that some of the learning (as a result of viewing the films) is apt to be superficial. This is an understandable and possible outcome because of the broad scope of the film series. The broad scope, however, does make the series suitable for either an introduction of new concepts or a review of previously learned concepts. In any case, a teacher should see the film before it is shown to the students so that he can set the stage for the film. At some points, the teacher may wish to stop the film and expand the explanation or answer questions. A follow-up discussion of the film would also be highly desirable. Any one of the films in this series is suitable for students in the ninth grade or higher, and the first two programs may be suitable for some classes in seventh and eighth grades. This reviewer found the films quite satisfactory for a college class in the teaching of arithmetic.

The film, *Fundamental Operations*, can be criticized for the abstract approach to the four basic operations. The meanings of addition and multiplication are not explained, and subtraction and division are presented only as inverse operations. The use of modular arithmetic is also questioned as a medium for understanding our rational number system and the fundamental operations with rational numbers. The lack of clarification of the meanings of number and the four operations with numbers causes a slight error in the film, *Fractions*. This error is that of giving "$\frac{1}{3}$ of 3" and "$3 \div 5$" as separate meanings of the fraction $\frac{3}{5}$, even though "$\frac{1}{3}$ of 3" is demonstrated by *dividing* 3 candy bars into 5 equal parts. This is a common misconception that can be forgiven, however.

In summary Dr. Jones makes an effective presentation in this series, especially when dealing with the historical development of the various topics. The most serious defect of the series in the opinion of the reviewer is that the series tries to accomplish too much.—*Richard D. Crumley.*

FILMSTRIPS

Amazing Arithmetic Set: Knowing Numbers; Fraction Facts; Fraction Forms; Some Sums; Different Differences; Pleasing Products; Quotient Quests; Fraction Findings; and *Zero —the Forgotten Number.* Eye Gate House, Inc., 2716 Forty-First Avenue, Long Island City 1, New York. 35 mm. filmstrips, color, with 6-page Teachers Manual; $4.00 for an individual filmstrip, $25.00 for complete set.

The first strip, *Knowing Numbers*, presents a review of the meaning of whole numbers and an introduction to fractions. The next seven strips deal with fractions and the addition, subtraction, multiplication, and division of fractions. The ninth strip explains the meaning and use of zero and presents ten problems involving zero. This set, written by Elizabeth Ragland, with drawings by Christine Brown, is designed for use in the intermediate grades and junior high school.

The photographic quality of these filmstrips is excellent and the presentation is interesting. The use of animals and people in the illustrations helps to bring about this general appeal. Definitions and rules are emphasized more than explanation and justification. This is partly due, of course, to the medium used. Many educators will object to the statement given in the strip *Fraction Facts* that " . . . the fraction $\frac{1}{2}$ means a number was broken into 2 parts, and 1 of the parts was used in this fraction." The reviewer finds this objectionable because a number is not broken into parts. Another weak spot was found in the strip *Pleasing Products*. One frame illustrates *multiplication by a mixed number* by using $1\frac{1}{2}$ as an addend ten times,

$$\begin{array}{r} 10 \\ \times 1\frac{1}{2} \\ \hline \end{array}$$

although the example is . No mention is made that this is possible because of the commutative law for multiplication.

This set of filmstrips can probably be best used as a review of the names for different kinds of fractions and as a review of the rules for processing fractions. Unless students have already acquired an understanding of the basic concepts concerning fractions and the processing of fractions, their seeing this set would help them only in memorizing the mechanical features connected with fractions.—*Richard D. Crumley.*

• WHAT IS GOING ON IN YOUR SCHOOL?

Edited by John A. Brown, University of Wisconsin, Madison 6, Wisconsin, and Houston T. Karnes, Louisiana State University, Baton Rouge 3, Louisiana

The mathematics program in West High School, Green Bay, Wisconsin

by Carl D. Roethig, West High School, Green Bay, Wisconsin

For months now, a lot of publicity has been given to mathematics through the daily and periodical press. Schools have been taken to task for inferior curriculums, their limited offerings, and negligence in stimulating larger and more extensive enrollment. Students have been berated for their indifference toward mathematics and for shying away from studying any further than actually required.

The following is a paragraph from an editorial in the local Green Bay newspaper, July 27, 1955:

> The National Society of Professional Engineers is alarmed at the lack of interest young people are showing in the study of mathematics. The society reports that only about 25 per cent of high school students are now studying mathematics in high school, probably only 12 per cent study geometry, and the interest in physics and chemistry is so small that many high schools do not have the courses.

Charges of this type might be a bit disturbing when they cast reflections on schools that promote relatively strong mathematics and science programs and that encourage as extensive an enrollment as is possible. No doubt there are many schools in our state where not only more students are enrolled in mathematics classes than the national average indicates, but who also know more mathematics.

We were particularly interested in determining what our situation in mathematics is in relation to the national picture in enrollment and achievement. Although standardized tests have been used by the mathematics department for many years, a complete compilation of the scores and relative data was never undertaken. Last year the usual tests were given, and scores were carefully tabulated and evaluated. Enrollments were checked and compared with previous years.

Green Bay public schools require two years of mathematics of all students for graduation. At their recent meeting the mathematics teachers, when questioned by the superintendent as to the advisability of continuing such a plan, voted unanimously in favor of retaining it. With mathematics again gaining in importance, many other systems may want to reinstate this requirement.

We lay no claims that our program is outstanding since it is still the traditional type, and other schools perhaps would be able to show a better record. But students here do display a considerable amount of interest in mathematics. Primarily this may be attributed to an excellent guidance program headed by competent counselors who respect the importance of the subject, and the willingness of teachers

and principal to make the necessary provisions for better working conditions and student success.

About 520 students, or almost 55 per cent of the 946 enrolled at West High School will be studying some form of mathematics in 1955–56. About 41 per cent of these mathematics students will be in the college-preparatory courses. The national average in 1952–53 for senior high schools, as found by Mr. Kenneth E. Brown[1] in his study, "Mathematics in Public High Schools," shows about 27 per cent of the students in some course of mathematics, and 32 per cent of the pupils enrolled in mathematics in grades 9–12 were in the non-college type courses. Our mathematics population in late years has shifted somewhat, but the major change is toward the college courses in the upper grades.

For eighteen or more years West has divided its geometry classes into either three or four levels so that the specially talented, the average, and the slow learners would be grouped in separate classes. When the range of the middle group is large, it is split into two sections, thus reducing the differences in individual abilities to a reasonable minimum. All levels are taught from the same outline, but the breadth and techniques vary.

Naturally the talented students can go faster and delve more deeply into the various aspects of the subject. The slow groups postulate some theorems after experimenting with the basic principles and facts related to them. A variety of models are used to study inductively the various situations, which are then followed by deductive proofs if not too involved. This leaves the middle group to the traditional course, using such devices as are necessary for proper comprehension of the required work. Models, experimental techniques, films, filmstrips, and field work are employed as special teaching tools for all levels to augment the learning process.

To reconfirm the effectiveness of this grouping procedure, which incidentally is based on criteria similar to that set forth in the Bulletin of the Secondary School Principals Association,[2] the Cooperative[3] and the Shaycoft[4] tests were administered. The results in percentile medians for the four levels, the talented to the slow respectively, are as follows: in Cooperative scores, 90, 88, 50, and 28; and in Shaycoft scores, 98, 94, 81, and 60. Almost half of the top group scored in the upper 10 percentile while about 23 per cent of the second, 7 per cent of the third, and none of the slow in the Cooperative scores attained that mark. In the Shaycoft test this tabulation ran in this order: 88 per cent, 67 per cent, 26 per cent, and 2 per cent. On the lower end of the scale neither of the two top groups scored below the sixtieth and thirtieth percentiles respectively, while the average had 25 per cent and the slow group had 52 per cent respectively in the bottom 30 percentile in the first test. In the second test all the scores of the best group were above the eightieth percentile, those of the second group were above the seventieth percentile, in the third above the thirtieth, and the slow group had about 4 per cent in the percentiles 11–20.

There seems to be a distinct advantage of one group over the other when these test results are compared. And since the failure list is only slightly more than 1 per cent, sectioning in this manner tends to be well worthwhile.

There are other values in this procedure, such as presenting proper challenges to each student that stimulate him to the limit of his capabilities, yet do not confront him with such insurmountable

[1] Kenneth E. Brown, "Mathematics in Public High Schools" (Washington, D.C.: U. S. Department of Health, Education and Welfare, Bulletin 1953, No. 5. U. S. Government Printing Office, 1953).

[2] "Mathematics in the Secondary School," The Bulletin of the National Association of Secondary Principals, 38: 203, May 1954. The National Association of Secondary Principals, 1201 Sixteenth St., Washington 6, D.C.

[3] *Cooperative*, Cooperative Test Service, 15 Amsterdam Avenue, New York, N.Y., 1940.

[4] *Shaycoft Plane Geometry Test* (Chicago: World Book Co., 1951).

obstacles that he gives up in despair. Generally the talented are inspired to elect other advanced courses, and even some of the slower students attempt them. About 26 per cent of the juniors study advanced algebra, and 12 per cent of the seniors study trigonometry and solid geometry. Mr. Brown's report shows an enrollment of 23 per cent for algebra and about 4 per cent for trigonometry in 1952–53.

There is good reason to believe that the sectioning done through the junior and senior high school aided students greatly in their achievement in subsequent advanced courses where no systematic grouping is employed. Advanced algebra students scored an 88 percentile median in the Blyth test[5] with almost 90 per cent of their scores in the 71–100 percentile range. The trigonometry classes, using Cooperative tests, scored 87 percentile median and had about 84 per cent in the 71–100 percentile range. The solid geometry median with Cooperative test was 93 in percentile score and had 80 per cent in the 71–100 range. We do not think that these scores could have been attained in our school without the aid of ability grouping. Teachers say that without this segregation it would be very difficult to scale the teaching techniques to the levels of the various students.

Some of this is likewise reflected in the non-college courses. Last year, at the beginning of the term, about 32 per cent of the tenth-grade general-mathematics students ranged from .5 to 3.2 grades below normal grade level with the majority of this group down by 1 to 2 years. About 1.5 per cent were 1 to 2 grades above their grade level, and only 3 students rated better than that. These levels were determined by the California Mathematics tests,[6] Advanced Forms, which were administered early in September and again late in May. Different forms were used for each group at the stated time. The final results indicated that 19 per cent were still below their new grade level (10.9), but 13 per cent made their mark, and almost 35 per cent gained from 2 to 4 grades. The median score was in the seventy-second percentile, and 23.3 per cent were below the national median.

The Snader General Mathematics Test[7] was given the final week with results matching the other scores rather closely. The median was in the 71.5 percentile and fewer than 20 per cent were below the national norm.

A one-semester consumers' mathematics course offered to seniors during each half of the year is elected by students who had general mathematics or were in the lower geometry sections. To these students we gave the Davis Test of Functional Competence[8] and the Ohio Senior Screening[9] with these results: the Davis test showed a 64 percentile median and the Ohio a 66 percentile median. In both the range was 9–95 in percentiles with two students below the twentieth.

Thus, it may not be too presumptuous to conclude that our students are putting forth considerable effort to secure an extensive mathematics background and that our guidance and sectioning contribute effectively toward stimulating interest in good subject choice and in achievement. These data also may indicate that our talented and slow learners have a distinct advantage over the heterogeneous grouping in that the former are unhampered in their progress and the latter learn on their own with greater meaning and without the usual lead of the brighter student. This setup could provide a springboard, when enrollment and talent permit, for

[5] Blyth, *Second-Year Algebra Test* (Chicago: World Book Co., 1954).

[6] *California Mathematics Test*, California Testing Service, 5916 Hollywood Blvd., Los Angeles 28, Cal., 1950.

[7] *Snader General Mathematics* (Chicago: World Book Co., 1951).

[8] David J. Davis, *Davis Test of Functional Competence in Mathematics* (Chicago: World Book Co., 1950).

[9] *Ohio Senior Screening*, Ohio Scholarship Tests, State Department of Education, State Office Bldg., Columbus 15, Ohio.

a specially gifted group to start an accelerated program in the junior high school that would permit them to follow through more advanced topics in senior high school than the conventional subjects normally include.

Unscramble, please!

by George Janicki, Elm School, Elmwood Park, Illinois

A slight variation of a mathematical word game is to try to unscramble the letters of a very familiar word to produce another word that is mathematical in nature.

The following list was composed by students in an eighth-grade class recently. The time limit is five minutes.

1. cleric
2. fete
3. tear
4. dad
5. neo-
6. tow
7. latitude
8. mite
9. once
10. mare
11. claimed
12. chin
13. dray
14. race
15. angel
16. girth
17. car
18. lien
19. net
20. squeal
21. cape
22. pest
23. lime
24. not
25. smile
26. anger
27. grin
28. signed
29. sue
30. raged

Calling Spring VII-MMMC

As an old traveler, I am indebted to paperbound thrillers,
Because you travel faster from Cleveland to Terre Haute when you travel with a lapful of victims and killers.
I am by now an authority on thumbprints and fingerprints and even kneeprints,
But there is one mystery I have never been able to solve in certain of my invaluable reprints.
I am happily agog over their funerals, which are always satisfactory followed by exhumerals,
But I can't understand why so many of them carry their copyright lines in Roman numerals.
I am just as learned as can be,
But if I want to find out when a book was first published, I have to move my lips and count on my fingers to translate Copyright MCMXXXIII into Copyright 1933.
I have a horrid suspicion
That something lies behind the publisher's display of erudition.
I may be oversensitive to clues,
But I detect a desire to obfuscate and confuse.
Do they think that because a customer cannot translate MCMXXXIII into 1933 because he is not a classical scholar,
He will therefore assume the book to have been first published yesterday and will therefore sooner lay down his XXV cents or I/IV of a dollar?
Or do they, straying equally far from the straight and narrow,
Think that the scholarly will snatch it because the Roman copyright line misleads him to believe it the work of Q. Horatius Flaccus or P. Virgilus Maro?
Because anybody can make a mistake when dealing with MCMs and XLVs and things, even Jupiter, ruler of gods and men;
All the time he was going around with IO he pronounced it Ten.

—*Ogden Nash*, The Private Dining Room.

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NCTM

THE NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS

Thirty-fourth Annual Meeting

Hotel Schroeder, Milwaukee, Wisconsin
April 11–14, 1956

Cooperation with other organizations having related interests will be a feature of the Thirty-fourth Annual Meeting of The National Council of Teachers of Mathematics at Milwaukee.

Robert M. Thrall, University of Michigan, Ann Arbor, Michigan, will make the opening general address on Thursday evening. John R. Mayor, past president of The National Council, and presently Director for the Science Teaching Improvement Program of the AAAS will address the college group Thursday afternoon.

William L. Duren, President of the Mathematical Association of America will speak on Saturday morning.

Fred A. Miller, Education Director of the Midwest Division of the National Association of Manufacturers will speak at a sectional meeting on Saturday afternoon. He will give practical suggestions as to how business and industry can assist the classroom teacher in mathematics.

At the banquet, Vincent Cushing of the Armour Research Foundation will discuss mathematics and guided missiles.

The National Council will cooperate with the Milwaukee State Teachers College by having the Saturday meeting of the Council combined with the Mathematics Conference which the college had tentatively scheduled for May. We are fortunate to have as the luncheon speaker that day, Dr. W. A. Brownell, Dean of the School of Education at the University of California at Berkeley and a recognized authority in the field of the teaching of arithmetic.

Rudolph E. Langer of the University of Wisconsin will speak on "Time Is Running Out!" on Friday morning. A host of outstanding leaders in the field of mathematics and mathematics education have accepted places on the sectional programs.

Social events of the convention will include sight-seeing trips on Wednesday and Thursday, a reception Thursday evening, banquet Friday evening, and a luncheon on Saturday.

Wednesday, April 11

Members of the Council fortunate enough to arrive in Milwaukee before Wednesday will be given the opportunity to visit schools in the Milwaukee area. Both public and private schools at all grade levels will welcome our members. The local chairman of school visits will appreciate hearing from persons who would like to have arrangements made for such visits.

Milwaukee is famous for its breweries. For those who are curious as to how they operate, a trip to one of these breweries has been arranged for Wednesday evening.

Thursday, April 12

The Delegate Assembly meets Thursday morning. Visitors as well as delegates are welcome. For those not attending the Assembly, sightseeing trips and the showing of mathematical films have been ar-

ranged. Some committee meetings are scheduled for this morning. Also, visits may be made to Milwaukee schools.

The afternoon program opens with a demonstration lesson at the secondary school level. This is followed by sectional meetings at all school levels and two special sections. Past and future research in mathematics education will be discussed by Maurice Hartung of Chicago University at the section sponsored by the Research Committee. Problems connected with the supervision of a mathematics program will be discussed by Jesse Osborn of Harris Teachers College, St. Louis, Missouri, and a panel with Martha Hildebrandt, past president of the Council, serving as chairman.

Thursday evening Dr. Thrall's address will be followed by the annual reception.

Friday, April 13

This will be a lucky day for mathematics teachers attending the convention. Dr. Langer will give the opening address in the morning. Outstanding section programs in the morning have been organized by grade level. In the afternoon the problems of integration, articulation, evaluation, curriculum, and teacher education will be discussed by a galaxy of leaders in the fields of mathematics and mathematics education.

Reports from officers and committee chairmen are scheduled for the late afternoon so that all members may know of the activities of the Council. At 4:30 the annual business meeting will be held.

The annual banquet, highlight of the convention, is scheduled for Friday evening, with Vincent Cushing as the banquet speaker.

Saturday, April 14

Since some teachers will not be able to attend the convention any day except Saturday, the program committee has arranged programs for all grade levels. There will be particular emphasis on the elementary school level, since we hope to have a large group of elementary school teachers attending that day. Dr. Brownell will speak at an elementary section program as well as at the luncheon. A panel will discuss equipping a classroom for maximum learning in arithmetic. Dr. E. T. McSwain of Northwestern University will discuss good teaching in arithmetic, and Dr. Lowry Harding of Ohio State University will discuss developing understandings of measurement. In addition to these, two discussion groups at the elementary school level have been arranged for Saturday afternoon.

Teaching by television, and methods in which business and industry can assist the classroom teacher of mathematics are also part of Saturday's program.

Dr. Duren will open the general session Saturday morning with a discussion of the study of the undergraduate program in the colleges of which he has been chairman for a number of years, and its relation to the curriculum at the elementary and secondary level.

The annual luncheon on Saturday promises to rival the banquet in importance as local committees for both events make elaborate plans. As has been stated, Dr. Brownell will be welcomed to the Middle West as the luncheon speaker.

The formal program does not close until 4:00 P.M. on Saturday. Members will want to make their hotel and transportation reservations accordingly.

Your professional dates

The information below gives the name, date, and place of meeting with the name and address of the person to whom you may write for further information. For information about other meetings, see the previous issues of THE MATHEMATICS TEACHER. Announcements for this column should be sent at least ten weeks early to the Executive Secretary, National Council of Teachers of Mathematics, 1201 Sixteenth Street, N.W., Washington 6, D. C.

NCTM convention dates

ANNUAL MEETING

April 11–14, 1956
Schroeder Hotel, Milwaukee, Wisconsin
Margaret Joseph, 1504 N. Prospect Avenue, Milwaukee 2, Wisconsin

JOINT MEETING WITH NEA

July 2, 1956
Portland, Oregon
Lesta Hoel, Public Schools, Portland, Oregon

SUMMER MEETING

August 19–22, 1956
University of California, Los Angeles, California
Clifford Bell, University of California, Los Angeles 24, California

CHRISTMAS MEETING

December 27–29, 1956
Arkansas State College, Jonesboro, Arkansas
Lyle J. Dixon, Arkansas State College, State College, Arkansas

Other professional dates

Western Sectional Meeting of the Illinois Council of Teachers of Mathematics

March 24, 1956
Canton Senior High School, Canton, Illinois
Albert Millington, Canton Senior High School, Canton, Illinois

Sixth Annual Conference for Teachers of Mathematics

April 20–21, 1956
Bowling Green University, Bowling Green, Ohio
Eugene Smith, University School, Columbus, Ohio

Women's Mathematics Club of Chicago and Vicinity

April 21, 1956
Mandel's Tea Room, State and Madison Streets, Chicago, Illinois
Jean Bryson, Waukegan Township High School, 425 Surf Street, Chicago 14, Illinois

Annual Spring Meeting of the Missouri Council of Teachers of Mathematics

April 21, 1956
Fontbonne College, St. Louis, Missouri
Margaret F. Willerding, Harris Teachers College, St. Louis, Missouri

Spring Conference of the Minnesota Council of Teachers of Mathematics

April 27–28, 1956
Coffman Memorial Union, University of Minnesota, Minneapolis, Minnesota
Frances S. Ek, 1134 Grand Avenue, St. Paul, Minnesota

Ninth Annual Conference on the Teaching of Elementary and Secondary Mathematics

April 28, 1956
Illinois State Normal University, Normal, Illinois
Clyde T. McCormick, Illinois State Normal University, Normal, Illinois

Seventh Annual Conference of the Michigan Council of Teachers of Mathematics

May 4–6, 1956
St. Mary's Lake Camp, Battle Creek, Michigan
Miss Murel Kilpatrick, 114 N. Hamilton Street, Ypsilanti, Michigan

Fifth Annual Meeting of the Pennsylvania Council of Teachers of Mathematics

May 5, 1956

University of Pittsburgh, Pittsburgh, Pennsylvania
Catherine A. V. Lyons, 12 S. Fremont Avenue, Pittsburgh 2, Pennsylvania

Chicago Elementary Teachers' Mathematics Club

May 14, 1956
Toffenetti's Restaurant, 65 W. Monroe Street, Chicago, Illinois
Anne T. Linehan, O'Toole School, Chicago, Illinois

Mathematics Institute

June 4–15, 1956
University of Oklahoma, Norman, Oklahoma
J. O. Hassler, Department of Mathematics, University of Oklahoma, Norman, Oklahoma

Seventh Annual Mathematics Institute of Louisiand State University

June 24–29, 1956
Louisiana State University, Baton Rouge, Louisiana
Houston T. Karnes, Louisiana State University, Baton Rouge, Louisiana

SUMMER INSTITUTE on SECONDARY MATHEMATICS

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June 18 to July 27

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Requests for details and application blanks should be addressed to Dr. H. Van Engen, Iowa State Teachers College, Cedar Falls, Iowa.

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For information and application blanks write to: Professor D. E. Richmond, Department of Mathematics, Williams College, Williamstown, Massachusetts.

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Are students throwing away money on "bargain" slide rules?

ROBERT JONES, *Manager of Educational Sales, Frederick Post Company*

A manufacturer points out that penny-wise buying of elementary slide rules is often wasteful.

Our purpose here is to suggest a slightly higher standard for elementary slide rules in high schools and colleges.

We know that a majority of students cannot afford expensive rules. We also know that the demands of most courses do not justify the cost of a professional rule.

Nevertheless, far too many students consider only *cost* and sacrifice performance, dependability and long life when buying their rules.

In visits to school stores throughout the country, we have seen countless "specials" selling for $.75 to $1.25. These prices are attractive, but the instruments are inadequate. They have the necessary scales, but a closer inspection reveals these weaknesses:

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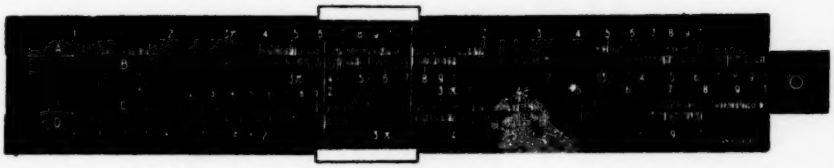

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